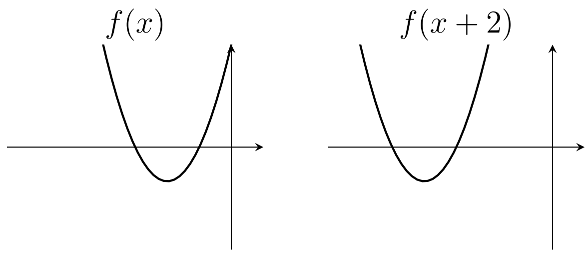
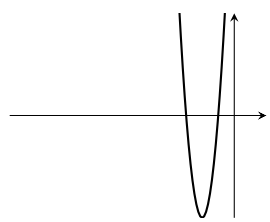
Graphs & Transformations – Solutions

Revision Questions

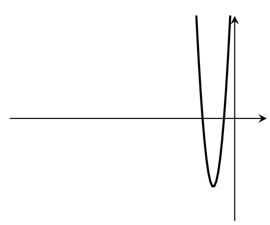
1. Note that $x^2 + 4x + 3 = (x+3)(x+1)$. The graph of $y = f(x+2)$ is the graph of $y = f(x)$ after it has been translated two units to the left.



For $y = 3f(2x)$, the graph is "squashed" by a factor of 2 parallel to the x-axis, then "stretched" by a factor of 3 parallel to the y-axis.



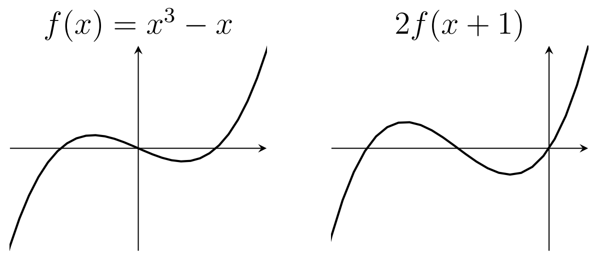
For $y = 2f(3x)$, the graph is squashed by a factor of 3 parallel to the x-axis, then stretched by a factor of 2 parallel to the y-axis



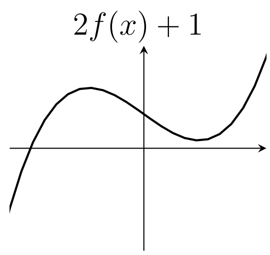
It's not the same as the previous graph. For example, the roots aren't in the same places.

$g(x) = x$ works for the last part; then $y = 20x$ in both cases.

2. The graph of $y = f(x)$ and the graph of $y = 2f(x+1)$;



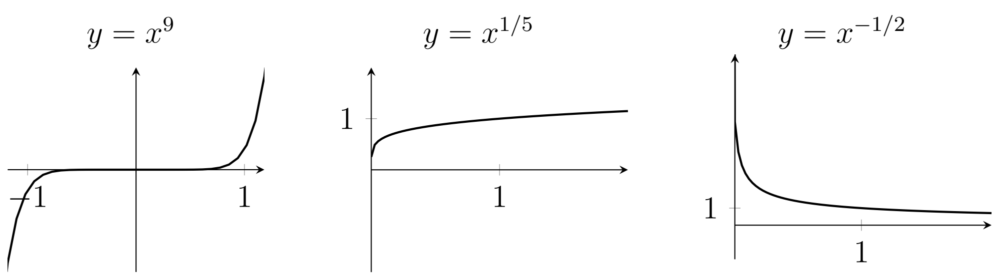
The graph of $y = 2f(x) + 1$;



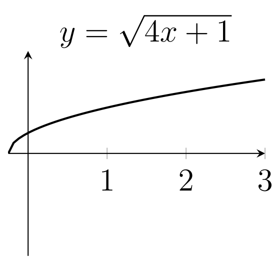
It's not the same as the previous graph. We could compare, for example, the values of $2f(x+1)$ and $2f(x)+1$ when $x = 0$.

$g(x) = x/3 + c$ for any constant c works for the last part.

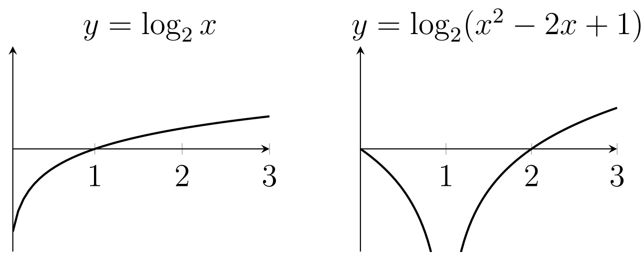
3. For large n, $y = x^n$ is close to zero between -1 and 1. For small positive $n < 1$, $y = x^n$ is approximately 1 between 0 and 1. For negative n, the graph increases to infinity near $x = 0$.



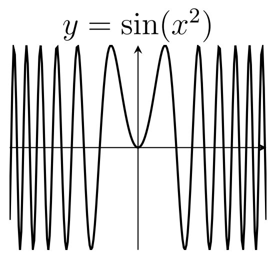
4. Note that $\sqrt{4x+1} = 2\sqrt{x + \frac{1}{4}}$ so this is a translation of the graph of $y = \sqrt{x}$ by $\frac{1}{4}$ units in the negative x-direction followed by a stretch parallel to the y-axis with scale factor 2.



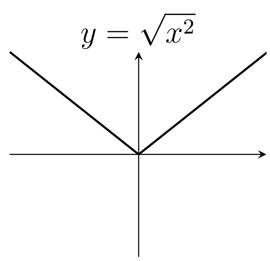
5. Note that $\log_2(x^2 - 2x + 1) = \log_2((x-1)^2)$.



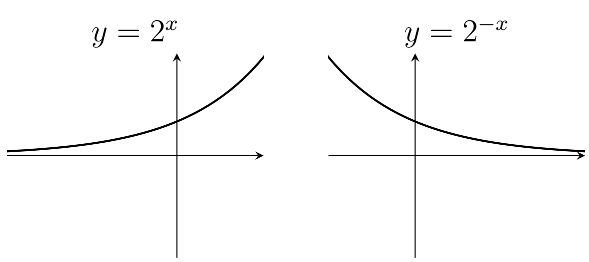
6. The function $\sin(x^2) = 0$ when $x^2 = 180°n$ for n a whole number, so the graph crosses the x-axis more and more frequently as x increases. The graph has reflectional symmetry in the y-axis.



7. If $x \geqslant 0$ then $\sqrt{x^2} = x$ but if $x < 0$ then $\sqrt{x^2} = -x$, because $\sqrt{u}$ is always the positive root.

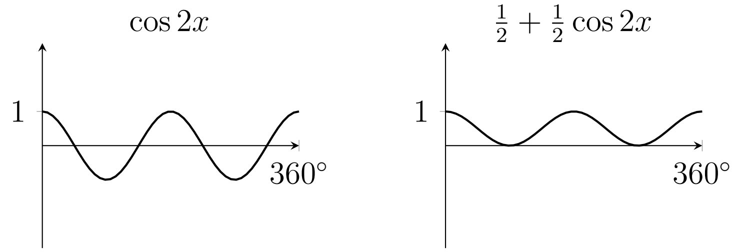


8. The graph of $y = 2^x$, and the graph of $y = 2^{-x}$.

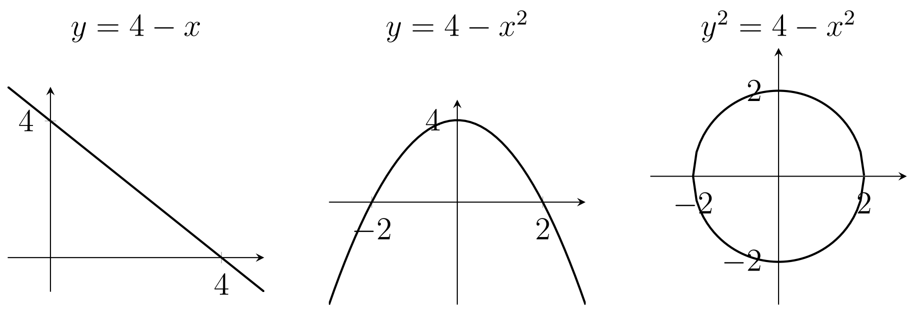


One is the reflection of the other, reflecting in the y-axis.

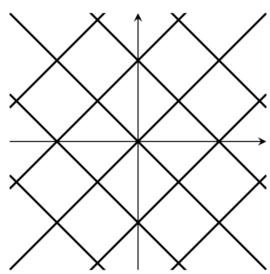
9. The graph of $y = \cos 2x$, and the graph of $y = \frac{1}{2} + \frac{1}{2}\cos 2x$.



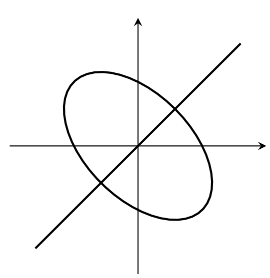
10. The equation $y = 4 - x$ is the equation of a straight line.
    The equation $y = 4 - x^2$ is the equation of a parabola.
    The equation $x^2 = 4 - y^2$ is the equation of a circle with radius 2.



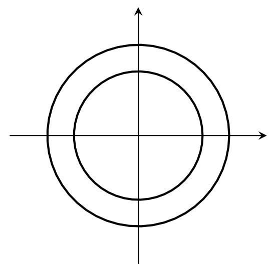
11. If $\cos x = \cos y$ then either $x = y$, or $x = y + 360°n$ for some whole number n, or $x + y = 360°n$ for some whole number n. These are the equations of lines.



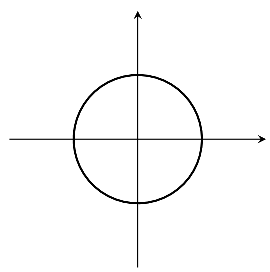
12. It's possible that $x = y$, but because $f(x) = x^3 - x$ is not a one-to-one function, there are other points (x, y) that we must consider. For some values of c there are three solutions to $x^3 - x = c$.



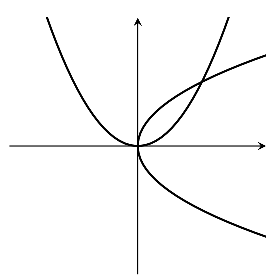
13. The equation $x^4+2x^2y^2+y^4-3x^2-3y^2+2=0$ simplifies to $(x^2+y^2)^2-3(x^2+y^2)+2=0$, which is a quadratic for x^2+y^2, with roots $x^2+y^2=1$ or $x^2+y^2=2$. This is a pair of circles.



14. The equation $x^6+3x^4y^2+3x^2y^4+y^6=1$ simplifies to $(x^2+y^2)^3=1$ so we have $x^2+y^2=1$ and this is a circle.



15. The equation rearranges to $x^3-xy=x^2y^2-y^3$. Take out a factor of x on the left and y^2 on the right to factorise this as $x(x^2-y)=y^2(x^2-y)$ so either $x^2=y$ or $x=y^2$. This is a pair of parabolas.



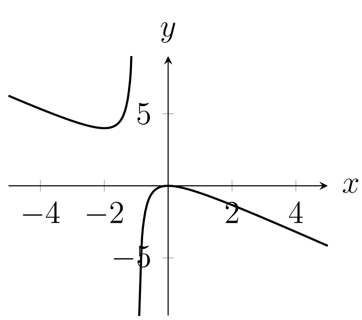
MAT Questions

MAT 2014 Q1I

- A translation parallel to the x-axis would give us something like $2^{(x-a)^2}$.
- A stretch parallel to the y-axis would give us something like $b \times 2^{x^2}$, and if we write $b = 2^c$ then this would be 2^{x^2+c}.
- If we complete the square, we find $x^2 - 4x + 3 = (x-2)^2 - 1$
- So
$$2^{x^2-4x+3} = 2^{(x-2)^2-1} = 2^{(x-2)^2} \times 2^{-1}.$$
- This is what we would get from a translation parallel to the x-axis followed by a stretch parallel to the y-axis.
- The answer is (b).

MAT 2017 Q1D

- The graph has been reflected in the x-axis and reflected in the y-axis to get from $y = f(x)$ to $y = -f(-x)$ (you might have used the phrase "stretch by a factor of -1", which means the same thing)
- Looking at the graph, there is a point at about $x = 1$ where the values of $f(x)$ are very large (either positive or negative), outside the range of this plot. So our transformed graph should have something similar around $x = -1$.
- Looking at the graph again, there are values of y between about -4 and 0 for which there's no graph (there are no values of x such that $y = f(x)$ for those values of y). After the reflections, this region will appear between $y = 0$ and about $y = 4$.
- The overall sort of shape of the graph won't be distorted by the reflections.
- My sketch looks something like this;



MAT 2008 Q3

(i) The graph $y = f(-x)$ represents a reflection in the y-axis.

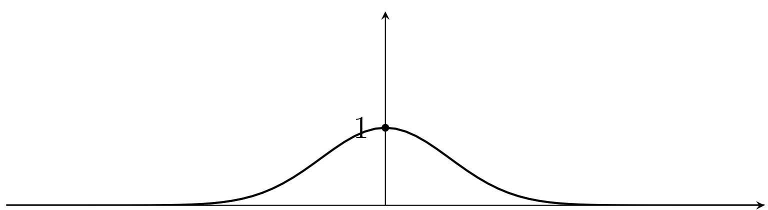
The graph $y = f(x-1)$ represents a translation by 1 unit in the positive x-direction.

The graph $y = -f(x)$ represents a reflection in the x-axis.

Trying these on the graph that we're given for $y = f(x)$, I get something that looks like (A) or (B) for the first and third transformations, while the graph (C) is clearly the translation. Looking at the axes, I can see that (A) is the reflection in the x-axis.

So $y = f(-x)$ is (B), $y = f(x-1)$ is (C), and $y = -f(x)$ is (A).

(ii) Note that $-x^2 \leqslant 0$, and gets very negative very quickly as x gets large (no matter whether $x > 0$ or $x < 0$). We also know that, if u is very negative, then 2^u is close to zero. The graph will have reflectional symmetry in the y-axis, and the y-intercept is 1.

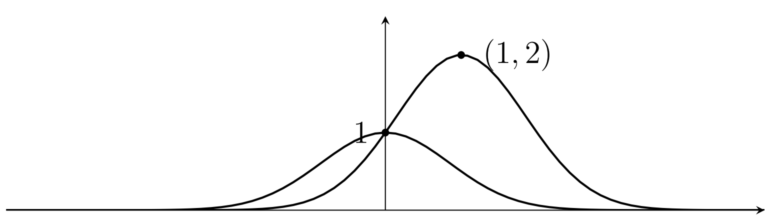


The maximum value of 2^{-x^2} comes where $-x^2$ is maximised, which happens when $x = 0$. The stationary point here has coordinates $(0, 1)$.

Now note that $2x - x^2 = -(x-1)^2 + 1$ if we complete the square. So

$$2^{2x-x^2} = 2 \times 2^{-(x-1)^2}$$

This is a translation by 1 unit in the positive x-direction, followed by a stretch parallel to the y-axis with scale factor 2. The y-intercept is 1 again, so the graphs intersect there. After the translations, the stationary point is now at $(1, 2)$. My sketch now looks like this;



(iii) Note that $2^{-(x-c)^2}$ is just a translation of 2^{-x^2}. We want the area under the graph between 0 and 1 to be large, so we want the big bit of the graph to be in that range. Let's park it right in the middle with $c = \frac{1}{2}$.

Extension

- Things that are unchanged; number of turning points, degree of the polynomial (if they're polynomials), and more.

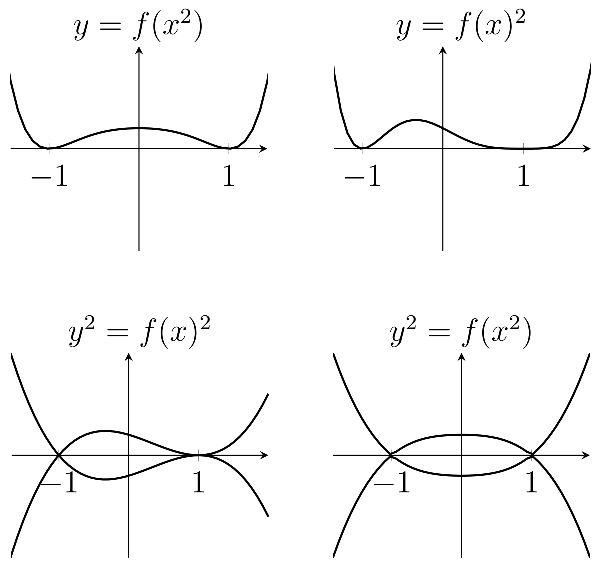
  Things that might change; number of roots, behaviour for large x, y-intercept, value at $x = 37$, and more.

- The graph of $f(x^2)$ just uses the part of the graph of $y = f(x)$ where $x \geqslant 0$, and there's some distortion. For this function $f(x)$, the values happen to be positive in $x \geqslant 0$. This graph will have reflectional symmetry in the y-axis.

  The graph of $y = f(x)^2$ takes values of $f(x)$ and squares them, so it'll definitely get a positive result. There's no obvious symmetry to this graph.

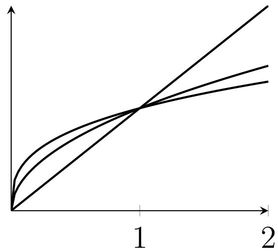
  If $y^2 = f(x)^2$ then either $y = f(x)$ or $y = -f(x)$. This graph will have reflectional symmetry in the x-axis.

  For $y^2 = f(x^2)$ we have to take our previous graph of $f(x^2)$ and imagine taking the square root of the values, and also the negative of that. This graph will have reflectional symmetry in both the x-axis and the y-axis.



MAT 2017 Q3

(i) We're being asked to sketch $y = x$ and $y = \sqrt{x}$ and $y = \sqrt[3]{x}$.



Note that all three graphs pass through the points $(0, 0)$ and $(1, 1)$.

(ii) The intersection points we found in the previous part are in fact intersection points for any pair of functions f_k and f_{k+1}. So the region we're looking at in this part lies between $x = 0$ and $x = 1$.

To calculate the area between $y = x^{1/k}$ and $y = x^{1/(k+1)}$, we'll need to integrate.

$$\int_0^1 x^{1/(k+1)} - x^k \,\mathrm{d}x = \left[\frac{x^{1+1/(k+1)}}{1+1/(k+1)} - \frac{x^{1+1/k}}{1+1/k}\right]_0^1 = \frac{k+1}{k+2} - \frac{k}{k+1}.$$

This fraction simplifies a bit, as follows;

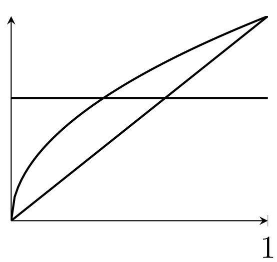
$$\frac{k+1}{k+2} - \frac{k}{k+1} = \frac{(k+1)^2 - k(k+2)}{(k+1)(k+2)} = \frac{1}{(k+1)(k+2)}.$$

The value when $k = 1$ is $\frac{1}{2\times 3}$, which is $\frac{1}{6}$.

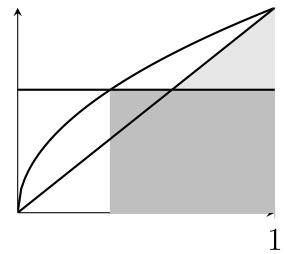
(iii) First we're looking for the point of intersection of $y = c$ with $y = x$. That's at (c, c).

Then the point of intersection of $y = c$ with $y = \sqrt{x}$ occurs when $x = c^2$, so the point is at (c^2, c).

(iv) The situation is as follows;



The region above the line $y = c$ looks easier to calculate, to me. I'll find that area by integrating $y = \sqrt{x}$ from $x = c^2$ to $x = 1$ and then subtracting a rectangle and a triangle.



That means that to find the area of the region above the line, I want

$$\int_{c^2}^{1} \sqrt{x}\,\mathrm{d}x - c(1-c^2) - \frac{1}{2}(1-c)^2$$

which is

$$\frac{2}{3}(1-c^3) - c + c^3 - \frac{1}{2} + c - \frac{1}{2}c^2.$$

In order for this to be half of the total region between f_1 and f_2, this area needs to be equal to $\frac{1}{12}$. Simplifying a bit, we want

$$\frac{1}{6} + \frac{1}{3}c^3 - \frac{1}{2}c^2 = \frac{1}{12}$$

Now multiply both sides by 12 and rearrange some more for $4c^3 - 6c^2 + 1 = 0$.

Finally, we're asked to find c. It's somewhere between 0 and 1. Let's try 1/2. That works!

There can't be any other solutions in that range, because as the line $y = c$ moves upwards, the area of the region above the line only decreases (so it can't be $\frac{1}{12}$ twice).

Extension

- These areas are easier to calculate if we reflect in the line $y = x$ by switching y and x. The curves become $y = x^k$ and $y = x^{k+1}$, and we're looking for the value of c such that the line $x = c$ splits the region between those curves into equal parts. So we just need

$$\int_0^c x^k - x^{k+1}\,\mathrm{d}x = \frac{1}{2(k+1)(k+2)}$$

where I've used the result from part (ii).

This simplifies to the given expression.