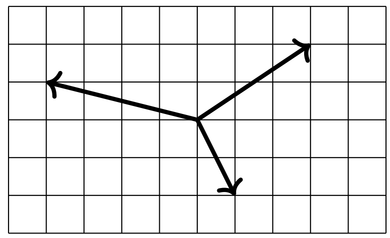
Geometry – Solutions

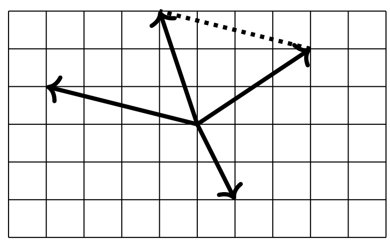
Revision Questions

1. Something like



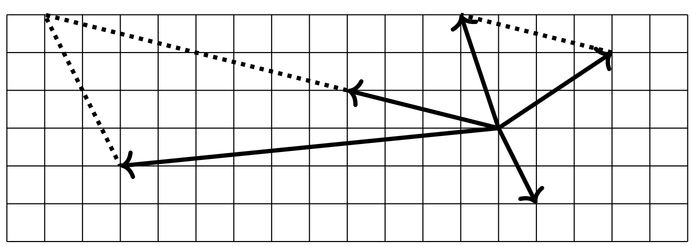
2. We add the components separately, so $\begin{pmatrix}3\\2\end{pmatrix} + \begin{pmatrix}-4\\1\end{pmatrix} = \begin{pmatrix}-1\\3\end{pmatrix}$.

   My diagram now looks like this.



3. You multiply a vector by a scalar by multiplying each component, so $3\begin{pmatrix}-4\\1\end{pmatrix} = \begin{pmatrix}-12\\3\end{pmatrix}$ and $2\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}2\\-4\end{pmatrix}$. Then add them together $\begin{pmatrix}-12\\3\end{pmatrix} + \begin{pmatrix}2\\-4\end{pmatrix} = \begin{pmatrix}-10\\-1\end{pmatrix}$.

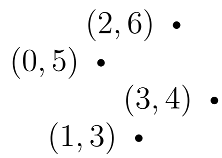
   My diagram now looks like this.



4. This line has gradient $(-1-5)/(3-1) = -3$ and goes through $(1,5)$ so it's $y-5 = -3(x-1)$ which can also be written as $y = 8-3x$.

5. This must be $y = 2x+c$ for some constant c, and the line goes through $(3,5)$ so $5 = 6+c$ and so the line is $y = 2x-1$.

6. I might try to show that all the sides are the same length, and that all the corners are right angles. First I need to draw a diagram to get the points in the right order.



   Now I can check that the distances from $(1,3)$ to $(3,4)$, from $(3,4)$ to $(2,6)$, from $(2,6)$ to $(0,5)$, and from $(0,5)$ to $(1,3)$ are all $\sqrt{5}$.

   To check the corners are right angles, I could check that the gradients of the lines for each side multiply to -1. Those gradients are all either $\frac{1}{2}$ or -2, so all the corners are right angles.

7. There are lots of examples that work! I decided to use the x-axis as one of my lines (that's $y = 0$), and then use something like $y = \sqrt{3} - ax$ and $y = \sqrt{3} + bx$ for some a and b; I've chosen those y-intercepts so that $(0, \sqrt{3})$ is a corner of the triangle.

   I need those two lines to go through $(\pm 1, 0)$. I can do that by choosing a and b carefully, and I end up with the three lines $y = 0$ and $y = \sqrt{3}(1 - x)$ and $y = \sqrt{3}(1 + x)$.

8. $(x+1)^2 + (y-2)^2 = 3^2$

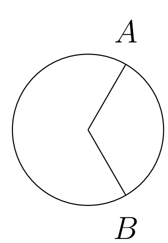
   The area is πr^2 and $r = 3$ so the area is 9π.

   The circle meets the x-axis where $(x+1)^2 + (0-2)^2 = 3^2$. That's $x = -1 \pm \sqrt{5}$.

   The circle meets the y-axis where $(0+1)^2 + (y-2)^2 = 3^2$. That's $y = 2 \pm \sqrt{8}$.

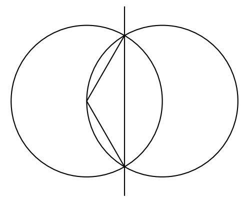
9. $x^2 + 9x + y^2 - 3y = \left(x + \frac{9}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 - \frac{81}{4} - \frac{9}{4}$. The equation of the circle is $\left(x + \frac{9}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = 10 + \frac{90}{4}$. So the centre is $\left(-\frac{9}{2}, \frac{3}{2}\right)$ and the radius is $\sqrt{\frac{65}{2}}$.

10. Draw a diagram.



   Since 120° is one-third of 360°, the length of the arc is one-third of the length of the circumference $2\pi r$ with $r = 2$. So the length of the arc is $\frac{4}{3}\pi$. The area is one-third of πr^2, which works out to be $\frac{4}{3}\pi$.

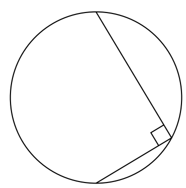
11. Draw a diagram.



Find the points of intersection. Taking the difference between the two equations gives $x^2 = (x-2)^2$, so $x = 2-x$ or $x = x-2$, which only has $x = 1$ as a solution. The y-coordinates are $\pm\sqrt{3}$, and the angle at the centre is $120°$. Let's aim to find the area to the right of $x = 1$ that's inside both circles. That's the area of the sector from the previous question, minus the area of a triangle. We can use $\frac{1}{2}ab\sin\theta$ to work out the area of the triangle, $\sqrt{3}$.

Then we'll need to double the area to get our final answer of $\frac{8}{3}\pi - 2\sqrt{3}$.

12. Draw a diagram.



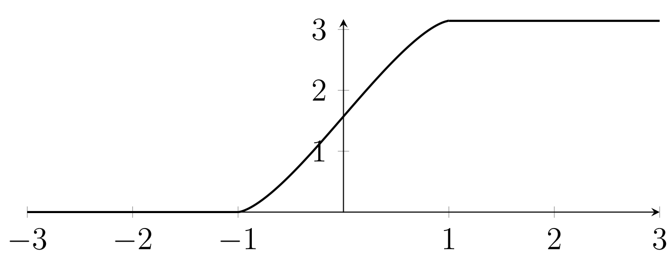
We could write down equations for the distance of a general point (x, y) to each of these points and set them equal to each other, but that's a lot of work.

Instead, note that the gradient of the line from $(0,0)$ to $(1,a)$ is a and the gradient of the line from $(1,a)$ to $(0, a+a^{-1})$ is $-a^{-1}$. These gradients multiply to -1, so the lines are at right-angles.

The angle in a semi-circle is a right-angle, so the line from the first point to the third point is the diameter of the circle.

The centre is at the midpoint of the diameter, so it's at $(0, \frac{1}{2}(a+a^{-1}))$.

13. The area $A(c)$ is zero if $c < -1$ and it's π if $c > 1$. In between, the area rises from 0 to π in a nice symmetric manner; slow then fast then slow.



MAT Questions

MAT 2016 Q1C

- The equation can be written as
$$\left(x+\frac{a}{2}\right)+\left(y+\frac{a}{2}\right)=c+\frac{a^2}{4}+\frac{b^2}{4}.$$
- The centre is at $\left(-\frac{a}{2},-\frac{b}{2}\right)$ and the radius is $\sqrt{c+\frac{a^2}{4}+\frac{b^2}{4}}$.
- The origin is in the circle if the distance to the centre is less than the radius
- This gives the inequality $\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}<\sqrt{c+\frac{a^2}{4}+\frac{b^2}{4}}$. Both sides are positive, so we want $\frac{a^2}{4}+\frac{b^2}{4}<c+\frac{a^2}{4}+\frac{b^2}{4}$, which happens only if $c>0$.
- Alternatively, substitute in $x=0$ and $y=0$ to discover that the origin lies on the boundary of the circle when $c=0$. The origin is outside the circle if c is very small, so the answer must be $c>0$.
- The answer is (a).

MAT 2017 Q1G

- No matter the value of θ, the point $(-1,1)$ is always on the line. As θ changes, the line rotates around, such that the line makes an angle of θ with the positive x-axis.
- The "larger" region is maximised if the line is at right angles to the radius between $(0,0)$ and $(-1,1)$.
- This happens for two values of θ, $135°$ and $315°$. (there are two values of θ corresponding to the same line, because replacing θ with $\theta+180°$ introduces a minus sign on *both* sides of the equation).
- The answer is (b).

MAT 2014 Q1D

- The line perpendicular to $y=mx$ that passes through $(1,0)$ is $y=-\frac{1}{m}(x-1)$.
- This line meets $y=mx$ when when $x=\dfrac{1}{1+m^2}$ and so $y=\dfrac{m}{1+m^2}$.
- We want the point on the "other side" of the line. Moving an equal distance will change the y-coordinate by the same amount again, that is, doubling it. So the point we're looking for has $y=\dfrac{2m}{1+m^2}$.
- The answer is (d).

MAT 2008 Q4

(i) Let's write down $(x-a)^2+(y-b)^2=r^2$ for the equation of the circle. The three points $(0,0)$ and $(p,0)$ and $(0,q)$ all lie on the circle, which gives three equations;

$$(0-a)^2+(0-b)^2=r^2, \qquad (p-a)^2+(0-b)^2=r^2, \qquad (0-a)^2+(q-b)^2=r^2.$$

We can simplify these a bit, and multiply out some squares, and take the difference between pairs of equations, to get

$$p^2-2pa=0, \qquad q^2-2qb=0, \qquad a^2+b^2=r^2$$

Now $p \neq 0$ and $q \neq 0$ so $a=\frac{p}{2}$ and $b=\frac{q}{2}$, and $r=\sqrt{\dfrac{p^2+q^2}{4}}$.

If we substitute those numbers into the equation $(x-a)^2+(y-b)^2=r^2$ and multiply out the squares, we get the equation in the question.

Along the way, we found the centre $(a,b)=\left(\frac{p}{2},\frac{q}{2}\right)$ and we found the radius. The area of C is $\pi\dfrac{p^2+q^2}{4}$.

(ii) We just found the area of the circle. The area of the triangle is $\frac{1}{2}pq$. If we write out the inequality in the question, we see that we're being asked to prove that

$$\pi\frac{p^2+q^2}{2pq} \geqslant \pi$$

for all positive real numbers p and q. This is true because $(p-q)^2 \geqslant 0$, and that rearranges to the inequality above (we can divide by pq because p and q are not zero). If I'm honest, I rearranged the equation first, factorised it as $(p-q)^2$, realised that was positive or zero, then presented all of that to you in the opposite order. Sometimes it's a good idea to work backwards as well as forwards... provided that your final argument makes sense, of course.

(iii) Now we're asked to solve

$$\pi\frac{p^2+q^2}{2pq}=2\pi$$

which rearranges to $p^2+q^2=4pq$. This is one equation for two variables, so we can't really solve it for p and q. But we just want expressions for the angles. From trigonometry, we know that q/p is $\tan\angle OPQ$, and something similar is true for $\tan\angle OQP$. That inspires me to divide the equation by p^2 and solve

$$1+\left(\frac{q}{p}\right)^2=4\left(\frac{q}{p}\right)$$

which is just a quadratic equation. The roots are $2\pm\sqrt{3}$ so the angles are $\tan^{-1}\left(2\pm\sqrt{3}\right)$.

Extension

- If $\tan\theta = 2-\sqrt{3}$ then

  $$\tan 2\theta = \frac{4-2\sqrt{3}}{1-\left(4-4\sqrt{3}+3\right)} = \frac{4-2\sqrt{3}}{4\sqrt{3}-6} = \frac{1}{\sqrt{3}}$$

  so 2θ is 30° (restricting to the range $0 \leqslant \theta \leqslant 180^\circ$). So θ must be 15°.

  If $\tan\theta = 2+\sqrt{3}$ then

  $$\tan 2\theta = \frac{4+2\sqrt{3}}{1-\left(4+4\sqrt{3}+3\right)} = \frac{4+2\sqrt{3}}{-4\sqrt{3}-6} = -\frac{1}{\sqrt{3}}$$

  so 2θ is 150° (restricting to the range $0 \leqslant \theta \leqslant 180^\circ$). So θ must be 75°.

- Differentiate to get $1-x^{-2}$ which is zero at $x=\pm 1$. The minimum value of the function occurs when $x=1$, and the value is 2.

  Alternatively, use the fact that $a^2+b^2 \geqslant 2ab$ with $a=x^{1/2}$ and $b=x^{-1/2}$. Then $x+x^{-1} \geqslant 2x^{1/2}x^{-1/2} = 2$.

MAT 2010 Q4

(i) If I drop a perpendicular from $(1, 2h)$ to $(1, 0)$ then I have a right-angled triangle with angle θ, opposite a side of length $2h$ and adjacent to a side of length $2h$. So $\tan\theta = 2h$.

(ii) $(1, 2h)$ lies in $x^2 + y^2 < 4$ if and only if $1 + 4h^2 < 4$, which rearranges to $h^2 < \frac{3}{4}$. Since $h > 0$, this condition is equivalent to $h < \sqrt{3}/2$.

(iii) The gradient of the line is $-h$ because the y-value changes by $-2h$ between $x = 1$ and $x = 3$. The line goes through $(3, 0)$ and has equation $y = -h(x - 3)$.

We could look for repeated roots between $x^2 + y^2 = 4$ and $y = -h(x - 3)$. In general if I substitute one into the other I get $x^2 + h^2(x - 3)^2 = 4$. If $h = 2/\sqrt{5}$ then this is $5x^2 + 4(x - 3)^2 = 20$. If we multiply this out, rearrange, and factorise, we get $(3x - 4)^2 = 0$ indicating that there is a double root at $x = 4/3$, so the line is tangent to the circle.

(iv) In this case, the point $(1, 2h)$ is outside the circle, because $\frac{2}{\sqrt{5}} > \frac{\sqrt{3}}{2}$ (I know this because $\frac{4}{5} > \frac{3}{4}$). The diagram is like the first picture in the question. The area inside both is the area of the sector with angle θ. So the area is $4\pi\dfrac{\theta}{360°}$ where $\tan\theta = 2h$.

(v) In this case, the point $(1, 2h)$ is inside the circle, because $\frac{6}{7} < \frac{\sqrt{3}}{2}$ (I know this because $\frac{36}{49} < \frac{3}{4}$ (I know this because $144 < 147$)). The diagram is like the second picture in the question.

Check that $(8/5, 6/5)$ lies on the circle; $(8/5)^2 + (6/5)^2 = (64/25) + (36/25) = 4$. Check that $(8/5, 6/5)$ lies on the line; $-\frac{6}{7}\left(\frac{8}{5} - 3\right) = \frac{6}{5}$.

The area is made up of a triangle with corners $(0, 0)$ and $(1, 12/7)$ and $(8/5, 6/5)$, plus a sector of a circle from the x-axis up to the line from the origin to $(8/5, 6/5)$.

Using the formula in the question, the area of the triangle is $\dfrac{27}{35}$. The area of the sector is $4\pi\dfrac{\alpha}{360°}$ where $\tan\alpha = \dfrac{6}{8}$. Add these.

Extension

- Let A be (a, b) and let C be (c, d) and let M be $(a, 0)$ and let N be $(c, 0)$. The area we want is the triangle OCN plus the trapezium $NCAM$ minus the triangle OAM. If we find all of these and simplify, we get $\frac{1}{2}(ad - bc)$. The absolute value signs come in because I haven't been very careful; I've assumed that (a, b) and (c, d) are a particular way around.
- Consider the cross product of the vectors $(a, b, 0) \times (c, d, 0) = (0, 0, ad - bc)$. We also know that the magnitude of $\mathbf{a} \times \mathbf{b}$ is $|\mathbf{a}||\mathbf{b}| \sin\theta$, which is (almost) exactly the formula for the area of the triangle. So we just need to take the magnitude of $\mathbf{a} \times \mathbf{b}$ and divide by 2 to get the area of the triangle.