Integration & Differentiation

MAT syllabus

Derivative of x^a , including for fractional exponents. Derivative of e^{kx}. Derivative of a sum of functions. Tangents and normals to graphs. Turning points. Second order derivatives. Maxima and minima. Increasing and decreasing functions. Differentiation from first principles. Indefinite integration as the reverse of differentiation. Definite integrals and the signed areas they represent. Integration of x^a (where $a \neq -1$) and sums thereof.

Revision

- The derivative of x^a is ax^{a-1}, including for fractional exponents like $a = \frac{1}{2}$.
- If k is a constant then the derivative of e^{kx} is ke^{kx}.
- If a is a constant then the derivative of $af(x)$ is a times the derivative of $f(x)$.
- The derivative of $y_1 + y_2$ is (the derivative of y_1)+(the derivative of y_2). Perhaps this looks too obvious to need stating, but remember that, for example, the square of $y_1 + y_2$ is not equal to (the square of y_1)+(the square of y_2).
- The tangent to a graph at a particular point is a line which has the same value and derivative as the graph at that point. So if we want the tangent to the graph $y = x^2$ at $x = 3$, we need the value of y (which is 9), and the value of the derivative (which is 6). The derivative of a line is its gradient, so we can write $y = 6x + c$ and solve for c using the value at $x = 3$ to get $y = 6x - 9$.
- The normal to a graph is a line which has the same value and is at right angles to the tangent. Two lines are at right angles if their gradients multiply to -1. So at the point above, we would want $y = -\frac{1}{6}x + c$ and, since the line goes through $(3, 9)$, we have $c = \frac{19}{2}$.
- If the derivative changes sign $(+/-)$ at a point, that's a turning point. You'll have zero derivative at the turning point, but that's not actually sufficient for the derivative to *change* sign (e.g. x^3 has zero derivative at $x = 0$, but that's not a turning point because the derivative is positive on both sides). A point with zero derivative is called a stationary point.
- The derivative of the derivative is called the second derivative. You can work out the derivatives one at a time. So the second derivative of x^a would be the derivative of ax^{a-1}, which is $a(a-1)x^{a-2}$. The second derivative of e^{kx} is k^2e^{kx}. The second derivative is the rate of change of the derivative.

- A turning point is a local maximum if the second derivative is negative at that point, or it's a local minimum if the second derivative is positive. "Maxima" is the plural of "maximum". "Minima" is the plural of "minimum". Overall, the function might have several local maxima, or none, and it might increase without bound (like $y = x$ for example) so just having derivative zero might not mean that that's biggest value of the function.

- If the derivative is positive, that's an increasing function. If it's negative, that's a decreasing function. In general a function might increase in some regions and decrease in other regions.

- If you have two points on a graph, you can join the line between them – that's called the chord. If you move the second point closer and closer to the first point, then the gradient of the chord gets closer and closer to the gradient of the tangent, which is the value of the derivative at that point. Calculating the gradient of the chord is a nice and sensible thing to do; it's just $\frac{y_2-y_1}{x_2-x_1}$, so this is called a "first principles" approach to differentiation.

- Indefinite integration (without limits as in $\int x^2 \, \mathrm{d}x$) is the reverse of differentiation in the sense that if the derivative of $f(x)$ is $g(x)$ then the indefinite integral of $g(x)$ is $f(x) + c$ where c could be any constant. You can use this to integrate any function which you could have got as the result of some differentiation.

- The integral of x^n is $\frac{x^{n+1}}{n+1}$, provided that $n \neq -1$.

- A definite integral (with limits as in $\int_1^2 x^2 \, \mathrm{d}x$) is written like $\int_a^b f(x) \, \mathrm{d}x$ where a and b are the two end-points. This is the difference in value of the indefinite integral at the two end-points; $F(b) - F(a)$ where the derivative of $F(x)$ is $f(x)$.

- If $f(x) > 0$ for $a < x < b$ then $\int_a^b f(x) \, \mathrm{d}x$ is the area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$.

- If $f(x) < 0$ for $a < x < b$ then $\int_a^b f(x) \, \mathrm{d}x$ is minus one times the area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$. Areas are supposed to be positive. The integral here is sometimes called the "signed area" to reflect the fact that it's got a minus sign.

- If $f(x)$ is sometimes positive and sometimes negative in $a < x < b$ then we can split into separate regions where $f(x)$ is positive or negative before applying the above.

- $\int_a^b f(x) \, \mathrm{d}x = -\int_b^a f(x) \, \mathrm{d}x$

- $\int_a^\infty f(x) \, \mathrm{d}x$ means the limit of $F(b) - F(a)$ for very large b (if this limit exists!). Formal knowledge of limits is not expected.

Revision Questions

1. Differentiate $x^{17} - x^{-17}$ with respect to x.

2. Differentiate $2\sqrt{x} + 3\sqrt[3]{x}$ with respect to x.

3. Differentiate $1 - e^{3x}$ with respect to x.

4. Find the tangent to the curve $y = e^x + x^2$ at $x = 2$.

5. Find the normal to the parabola $y = x^2$ at $x = 3$.

6. Find the turning points of the curve $y = x^4 - 2x^3 + x^2$. Identify whether the turning points are maxima or minima. For which values of x is $y = x^4 - 2x^3 + x^2$ increasing? For which values of x is it decreasing?

7. Two points A and B are on the curve $y = x^3 + x^2 + x + 1$. A is fixed at $(1, 4)$. The point B moves along the curve towards A. What happens to the line through A and B?

8. Find the area enclosed between the polynomial $y = x^2 + 4x + 3$ and the x-axis.

9. Find
$$\int \frac{x+3}{x^3}\,\mathrm{d}x, \qquad \int \sqrt[3]{x}\,\mathrm{d}x, \qquad \int \left(\left(x^2\right)^3\right)^5 \mathrm{d}x, \qquad \int \left(x^2+1\right)^3 \mathrm{d}x$$

10. By thinking about the area that the integral represents, explain why
$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \int_{-1}^{1} f(-x)\,\mathrm{d}x.$$

11. Let $I_1 = \displaystyle\int_1^{10} \frac{1}{x}\,\mathrm{d}x$ and let $I_2 = \displaystyle\int_{10}^{100} \frac{1}{x}\,\mathrm{d}x$ (you are not expected to calculate either of these integrals). By considering a rescaling of the graph $y = \dfrac{1}{x}$, and the area under that graph, prove that $I_1 = I_2$. Deduce that $\displaystyle\int_1^N \frac{1}{x}\,\mathrm{d}x$ with $N > 1$ can be made arbitrarily large by increasing N.

12. Let $I_3 = \displaystyle\int_1^3 \frac{1}{1+x^2}\,\mathrm{d}x$. Let $I_4 = \displaystyle\int_1^3 \frac{x^2}{1+x^2}\,\mathrm{d}x$. Without calculating either integral, write down a relationship between I_3 and I_4.

13. Calculate $\displaystyle\int_1^3 \frac{x^4}{1+x^2}\,\mathrm{d}x$ in terms of I_3 and/or I_4 from the previous question.

MAT questions

MAT 2017 Q1A

Let

$$f(x) = 2x^3 - kx^2 + 2x - k.$$

For what values of the real number k does the graph $y = f(x)$ have two distinct real stationary points?

(a) $-2\sqrt{3} < k < 2\sqrt{3}$

(b) $k < -2\sqrt{3}$ or $2\sqrt{3} < k$

(c) $k < -\sqrt{21} - 3$ or $\sqrt{21} - 3 < k$

(d) $-\sqrt{21} - 3 < k < \sqrt{21} - 3$

(e) all values of k.

[See the next page for hints]

MAT 2018 Q1A

The area of the region bounded by the curve $y = \sqrt{x}$, the line $y = x - 2$ and the x-axis equals

(a) 2, (b) $\frac{5}{2}$, (c) 3, (d) $\frac{10}{3}$, (e) $\frac{16}{3}$.

[See the next page for hints]

MAT 2018 Q1G

The parabolas with equations $y = x^2 + c$ and $y^2 = x$ touch (that is, meet tangentially) at a single point. It follows that c equals

(a) $\frac{1}{2\sqrt{3}}$, (b) $\frac{3}{4\sqrt[3]{4}}$, (c) $-\frac{1}{2}$, (d) $\sqrt{5} - \sqrt{3}$, (e) $\sqrt{\frac{2}{3}}$.

[See the next page for hints]

Hints
MAT 2017 Q1A

- "Stationary points" makes me think of differentiation. What's the derivative of $f(x)$? What's that got to do with stationary points?
- In general, it's hard to decide how many real roots a polynomial has. For polynomials with low degree though, you have some experience!

MAT 2018 Q1A

- First, draw a sketch of the curve and the line and the x-axis, including any points of intersection.
- Sometimes it's helpful to break up an area into smaller sections and tackle each section separately.
- We could integrate $\sqrt{x}$ from 0 to 4 to find an area. Which area would that be?

MAT 2018 Q1G

- We have two separate unknowns; we don't know the value of c, and we don't know the value of x at the point where the parabolas touch.
- Note that if $y^2 = x$ then $y = \pm\sqrt{x}$. Which part of the curve do we want?
- If the curves meet tangentially, then they have the same tangent at that point, so they have the same derivative at that point.
- You might get two equations to solve, one which involves c and one which doesn't. Substituting one equation into the other might be the way to go.

MAT 2009 Q3

For a positive whole number n, the function $f_n(x)$ is defined by

$$f_n(x) = \left(x^{2n-1} - 1\right)^2.$$

(i) Sketch the graph of $y = f_2(x)$ labelling where the graph meets the axes.

(ii) On the same axes sketch the graph of $y = f_n(x)$ where n is a large positive integer.

(iii) Determine

$$\int_0^1 f_n(x)\,\mathrm{d}x.$$

(iv) The *positive* constants A and B are such that

$$\int_0^1 f_n(x)\,\mathrm{d}x \leqslant 1 - \frac{A}{n+B} \quad \text{for all } n \geqslant 1.$$

Show that

$$(3n-1)(n+B) \geqslant A(4n-1)\,n,$$

and explain why $A \leqslant 3/4$.

(v) When $A = 3/4$, what is the smallest possible value of B?

[See the next page for hints]

Hints

(i) We have $f_2(x) = (x^3 - 1)^2$. Where are the intercepts? What happens when x is very large?

(ii) First, think about x^{2n-1} for large n, in different cases depending on whether x is large, small, positive, negative, zero.

(iii) Just like we did for the other MAT question, it's probably best to multiply out the square here and integrate term-by-term.

(iv) You've got an expression for the left-hand side. This is supposed to be true for all n. To get your head around that, imagine plugging in different values of n, including some when n is small / large.

(v) Substitute $A = 3/4$ and expand both sides of the inequality. Rearrange to get everything on one side. Could this inequality really be true for all positive whole numbers n? How?

Extension

[Just for fun, not part of the MAT question]

- What happens if, instead of being a positive whole number, $n = \frac{1}{2}$?
- The expression that you've found for $\int_0^1 f(x)\,\mathrm{d}x$ holds for any real number n, provided that $n > n_{\max}$ where c is some real number. Find the largest real value of n for which your expression does not hold, and explain why it doesn't hold for any smaller value of n.
- Sketch

$$y = 1 - \frac{3x-1}{x(4x-1)} \quad \text{for} \quad x > n_{\max}$$

MAT 2015 Q3 (modified)

We define $|x|$ to be equal to x if $x \geqslant 0$ and to $-x$ if $x < 0$.

We say that a function f is a *good approximation* to a function g if

$$|f(x) - g(x)| \leqslant \frac{1}{100} \qquad \text{whenever} \qquad 0 \leqslant x \leqslant \frac{1}{2}.$$

We say that a function f is an *excellent approximation* to a function g if

$$|f(x) - g(x)| \leqslant \frac{1}{320} \qquad \text{whenever} \qquad 0 \leqslant x \leqslant \frac{1}{2}.$$

(i) Give an example of two functions f and g such that f is a good approximation to g but f is not an excellent approximation to g.

(ii) Show that if

$$f(x) = x \qquad \text{and} \qquad g(x) = x + \frac{\sin(4x^2)}{400}$$

then f is an excellent approximation to g.

For the remainder of the question, let $f(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$.

(iii) Show that if

$$g(x) = 1 + \int_0^x f(t)\,\mathrm{d}t,$$

then f is an excellent approximation to g.

Now consider (but do not attempt to solve for) the function $h(x)$ which satisfies

$$h(x) = 1 + \int_0^x h(t)\,\mathrm{d}t \qquad \text{whenever} \qquad x \geqslant 0.$$

(iv) Show that for $x \geqslant 0$

$$h(x) - f(x) = g(x) - f(x) + \int_0^x (h(t) - f(t))\mathrm{d}t.$$

(v) You are given that $h(x) - f(x)$ has a maximum value on the interval $0 \leqslant x \leqslant \frac{1}{2}$ at $x = x_0$. Explain why

$$\int_0^x (h(t) - f(t))\,\mathrm{d}t \leqslant \frac{1}{2}(h(x_0) - f(x_0)) \qquad \text{whenever} \qquad 0 \leqslant x \leqslant \frac{1}{2}.$$

(vi) You are also given that $f(x) \leqslant h(x)$ for all $0 \leqslant x \leqslant \frac{1}{2}$. Show that f is a good approximation to h when $0 \leqslant x \leqslant \frac{1}{2}$.

[See the next page for hints]

Hints

(i) Functions can be simple, if you like. What do you need, in words?

(ii) What's $f(x) - g(x)$ equal to here? Remember what you know about $\sin(x)$ in the context of inequalities.

(iii) Calculate $g(x)$ directly from the definition of $f(x)$.

For an excellent approximation, we once again want to look at $f(x) - g(x)$.

Remember that $|x| = -x$ if $x < 0$.

Remember that we're only looking at $0 \leqslant x \leqslant \frac{1}{2}$ in the definition of an excellent approximation.

(iv) We have a strange equation for $h(x)$. Are there any similarities between that equation and the one that we're trying to show? Remember that $g(x)$ was also defined in terms of a strange integral; can you see that here too?

There's an $f(x)$ on each side; we could ignore that to start with and then, as a final step, subtract $f(x)$ from each side.

(v) The integral of a function from 0 to x represents an area, and if we know that the function is bounded by a maximum value, then the area is bounded by some theoretical maximum area. Given that "height restriction", which function would maximise the area? And what would the area be?

(vi) We proved a strange equality in part (iv) that we haven't used yet. That makes me think that we should try to use it now in this last part.

We've proved that the integral in that strange inequality is bounded.

As part of the definition of a good approximation, we need to show that

$$h(x) - f(x) \leqslant \frac{1}{100}$$

whenever $0 \leqslant x \leqslant \frac{1}{2}$. To check that, it would be enough to check that the maximum value of $h(x) - f(x)$ is less than $1/100$.

Extension

[Just for fun, not part of the MAT question]

- Check that $h(x) = e^x$ satisfies

$$h(x) = 1 + \int_0^x h(t)\,\mathrm{d}t.$$

- Find an example of a function $h(x)$ that satisfies

$$h(x) = 2 + \int_0^x 3h(t)\,\mathrm{d}t.$$