Geometry

MAT syllabus

Co-ordinate geometry and vectors in the plane. The equations of straight lines and circles. Basic properties of circles. Lengths of arcs of circles.

Revision

- Points in the plane can be described with two co-ordinates (x, y). The x-axis is the line $y = 0$, and the y-axis is the line $x = 0$.
- A vector $\begin{pmatrix} x \\ y \end{pmatrix}$ can store the same information as a pair of co-ordinates. Used in that sense, the vector is called a position vector.
- A vector can also describe the displacement from one point to another, so that $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ could represent the displacement from $(1, 1)$ to $(3, 2)$ for example.
- Vectors can be added by adding the components separately. To show that in a diagram, we might interpret the first vector as a position vector (drawing an arrow starting from the origin) and then interpret the second as a displacement (drawing an arrow starting from the end of the first vector).
- The magnitude of the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is $\sqrt{x^2 + y^2}$.
- The distance from A to B is the magnitude of the vector displacement from A to B. The distance from (x_1, y_1) to (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- A vector can be multiplied by a number by multiplying each component by that number. The result is a vector in the same direction but with scaled magnitude.
- A straight line has equation $y = mx+c$, where m is the gradient and c is the y-intercept. Other ways to write the equation of a line are $ax + by + c = 0$ (where that's a different c to the one in the previous expression) or $y - y_1 = m(x - x_1)$. The last expression is useful because that line goes through the point (x_1, y_1) and has gradient m, which might be information that we've been given.
- Two lines are parallel if and only if they have the same gradient. Two lines are perpendicular if and only if their gradients multiply to give -1.
- The equation of the circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.
- The angle in a semicircle is a right angle; if AB is the diameter of a circle, and C is on the circle, then $\angle ACB = 90°$.

- The tangent is at right angles to the radius at any point on a circle's circumference.
- A circle with radius r has area πr^2 and circumference $2\pi r$.
- If two radii of a circle of radius r make an angle of $\theta < 180°$ (in degrees), then the length of the minor arc between those radii is $\frac{\theta}{360°}2\pi r$. The area of the sector enclosed by that arc and the radii is $\frac{\theta}{360°}\pi r^2$.

Revision Questions

1. Draw a diagram to show the three separate position vectors $\begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}-4\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-2\end{pmatrix}$.
2. Add the vectors $\begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}-4\\1\end{pmatrix}$. Show this on your diagram.
3. Find $3\begin{pmatrix}-4\\1\end{pmatrix} + 2\begin{pmatrix}1\\-2\end{pmatrix}$. Show this on your diagram.
4. Find the equation of the line through the points $(1, 5)$ and $(3, -1)$.
5. Find the equation of the line through the point $(3, 5)$ with gradient 2.
6. Show that the points $(0, 5)$, $(1, 3)$, $(2, 6)$, and $(3, 4)$ lie on the corners of a square.
7. Find equations of three lines such that the finite region bounded by the three lines is an equilateral triangle.
8. A circle has centre $(-1, 2)$ and radius 3. Write down an equation for the circle. What's the area of this circle? Where does this circle cross the axes?
9. A circle is given by $x^2 + 9x + y^2 - 3y = 10$. Find the centre and radius of the circle.
10. Points A and B lie on a circle with centre O and radius 2. The angle $\angle AOB$ is 120°. Find the length of the arc between A and B. Find the area enclosed by that arc and the radii OA and OB.
11. Two circles are given by $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$. Find the area of the region that's inside both circles.
12. The points $(0, 0)$ and $(1, a)$ and $(0, a + a^{-1})$ all lie on the same circle. Find the centre of the circle in terms of a.
13. A circle has centre $(c, 0)$ and radius 1. The area in the region $x > 0$ which is inside the circle depends on c, and we'll call it $A(c)$. Sketch a graph of $A(c)$ against c.

MAT Questions

MAT 2016 Q1C

The origin lies inside the circle with equation

$$x^2 + ax + y^2 + by = c$$

precisely when

(a) $c > 0$, (b) $a^2 + b^2 > c$, (c) $a^2 + b^2 < c$, (d) $a^2 + b^2 > 4c$, (e) $a^2 + b^2 < 4c$.

[See the next page for hints]

MAT 2017 Q1G

For all θ in the range $0 \leqslant \theta < 360°$ the line

$$(y-1)\cos\theta = (x+1)\sin\theta$$

divides the disc $x^2 + y^2 \leqslant 4$ into two regions. Let $A(\theta)$ denote the area of the larger region.

Then $A(\theta)$ achieves its maximum value at

(a) one value of θ , (b) two values of θ, (c) three values of θ,
(d) four values of θ, (e) all values of θ .

[See the next page for hints]

MAT 2014 Q1D

The reflection of the point $(1, 0)$ in the line $y = mx$ has coordinates

(a) $\left(\dfrac{m^2+1}{m^2-1}, \dfrac{m}{m^2-1}\right)$, (b) $(1, m)$, (c) $(1-m, m)$,
(d) $\left(\dfrac{1-m^2}{1+m^2}, \dfrac{2m}{1+m^2}\right)$, (e) $\left(1-m^2, m\right)$.

[See the next page for hints]

Hints

MAT 2016 Q1C

- The circle $x^2 + ax + y^2 + by = c$ is written in quite an unusual way. Where is the centre of this circle? What's the radius?
- Given the location of the centre and the radius, how can you check whether another point is inside the circle?
- If you use the Pythagorean theorem for a distance, be careful with inequalities.

MAT 2017 Q1G

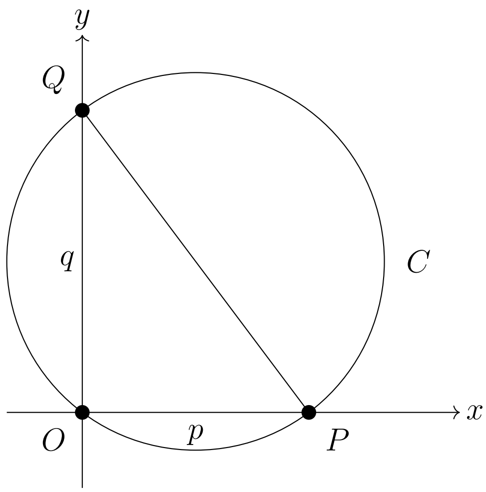
- The line $(y - 1)\cos\theta = (x + 1)\sin\theta$ is written in quite an unusual way. Can you find any points that lie on the line?
- Are there any points (x, y) that lie on the line for all values of θ?
- What happens as θ changes? Sketch some special cases as θ changes from 0 to 360°.
- $A(\theta)$ is the area of the "larger" region. Is this region always on the same side of the line?

MAT 2014 Q1D

- The line segment between $(1, 0)$ and the reflection of that point should meet the line $y = mx$ at right angles, and the midpoint of the line segment should lie on $y = mx$.
- Two lines are at right angles if their gradients multiply to -1.
- If you draw a diagram, you might spot a pair of congruent triangles. Or you could set up some algebra to express the idea that the midpoint lies on the line $y = mx$.

MAT 2008 Q4

Let p and q be positive real numbers. Let P denote the point $(p, 0)$ and let Q denote the point $(0, q)$.



(i) Show that the equation of the circle C which passes through P, Q, and the origin O is

$$x^2 - px + y^2 - qy = 0.$$

Find the centre and area of C.

(ii) Show that

$$\frac{\text{area of circle } C}{\text{area of triangle } OPQ} \geqslant \pi$$

(iii) Find expressions for the angles OPQ and OQP if

$$\frac{\text{area of circle } C}{\text{area of triangle } OPQ} = 2\pi$$

[See the next page for hints]

Hints

(i) My strategy for this part is to solve in the opposite order; I'll write down an equation for the circle which I'm happy with, and then I'll make it look like the one in the question.

In general the equation for a circle is $(x-a)^2+(y-b)^2=r^2$. We can plug in the points that we know lie on the circle and we'll get equations for a and b and r.

To find the centre and radius, I can either rearrange the expression in the question to make it look more like a familiar equation for a circle. Or I could use some of my working above.

Alternatively, draw a right-angle on your copy of the diagram and recall a geometric fact about circles.

(ii) We know expressions for both of these areas in terms of p and q. Then we've got an inequality to prove.

To prove an inequality like $a^2+b^2 \geqslant 2ab$, we might move all the terms to one side and try to spot a square (because squares are positive or zero).

(iii) The inequality is now an equality! Time to solve for p and q... except we can't. Not entirely, that is. There's a bit of ambiguity because if we double p and also double q then both areas will go up by a factor of 4, so the equality will still hold. At best, we can work out the ratio between p and q. This is enough to find the angles in terms of inverse trigonometric functions (can you see why?)

There's another ambiguity; we don't know which way round P and Q are (which one is larger? We don't know). Hopefully, we'll get an equation with at least two solutions.

Extension

[Just for fun, not part of the MAT question]

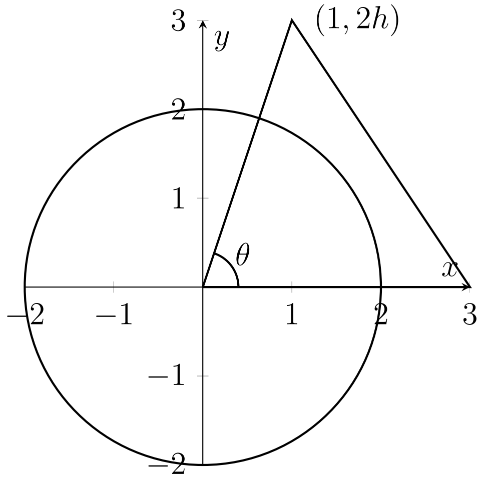
- Here's an equation for $\tan 2\theta$ (not on the MAT syllabus).

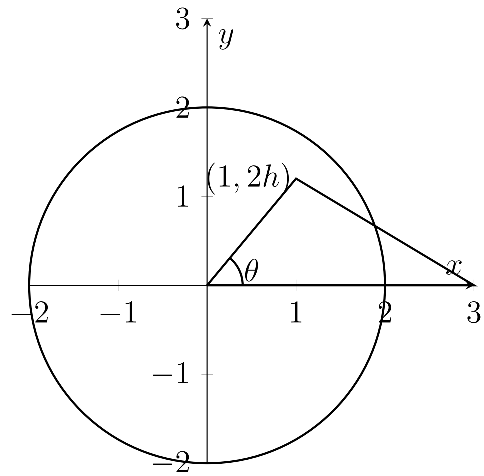
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

  Set θ to be the value of angle OPQ you found in this question, for which you know the value(s) of $\tan\theta$. Calculate $\tan 2\theta$. Deduce the exact value(s) of θ in degrees.

- Find the minimum value of $x+x^{-1}$ for $x>0$.

MAT 2010 Q4 (modified)





The three corners of a triangle T are $(0,0)$, $(3,0)$, $(1,2h)$ where $h > 0$. The circle C has equation $x^2 + y^2 = 4$. The angle of the triangle at the origin is denoted as θ. The circle and triangle are drawn in the diagrams above for different values of h.

(i) Express $\tan\theta$ in terms of h.

(ii) Show that the point $(1,2h)$ lies inside C when $h < \sqrt{3}/2$.

(iii) Find the equation of the line connecting $(3,0)$ and $(1,2h)$.
Show that this line is tangential to the circle C when $h = 2/\sqrt{5}$.

(iv) Suppose now that $h > 2/\sqrt{5}$. Find the area of the region inside both C and T in terms of θ.

(v) Now let $h = 6/7$. Show that the point $(8/5, 6/5)$ lies on both the line (from part (iii)) and the circle C.

Hence show that the area of the region inside both C and T equals

$$\frac{27}{35} + \frac{\alpha\pi}{90^\circ}$$

where α is an angle in degrees whose tangent, $\tan\alpha$, you should determine.

[You may use the fact that the area of a triangle with corners at $(0,0)$, (a,b), (c,d) equals $\frac{1}{2}|ad - bc|$.]

[See the next page for hints]

Hints

(i) Drop a perpendicular line from $(1, 2h)$ to the x-axis. Your equation for $\tan\theta$ should be nice and simple.

(ii) Points inside the circle have $x^2 + y^2 < 4$, because the distance from such a point to the origin is less than 2. For this question, this is an inequality involving h. Try to rearrange it for h.

(iii) Careful; the point of tangency is not $(1, 2h)$. To be tangential, we would need a single point which is on the line and also on the circle $x^2 + y^2 = 4$.

(iv) In order to get our picture right, we'll need to know whether that point is inside or outside the circle. A good way to compare numbers like $\sqrt{a}/b$ and $c/\sqrt{d}$ is to compare their squares.

Remember that we know the area of a sector of a circle.

(v) This case is different from the previous part. Again, check whether the point $(1, 2h)$ lies inside the circle using part (ii). We'll need to compare some square roots again.

The question gives us the coordinates of a point where the line crosses the circle. Mark this on your diagram.

The part of the answer that's a rational number could come from the area of a triangle using the hint at the end of the question. The part of the answer that involves angles and π is related to the area of a circle. We might guess that an expression like this comes from the area of a triangle plus the area of a sector.

Extension

[Just for fun, not part of the MAT question]

- If you've learned about the area of a trapezium, drop perpendiculars from (a, b) and (c, d) to the x-axis, identify the areas of two right-angled triangles and one trapezium, and deduce the fact about triangle area given in this question.
- If you've learned about the vector product (also known as the cross product), explain the fact about triangle area given in this question.