

Logarithms and powers – Solutions

Revision Questions

- $(2^3)^4 = 2^{12}$. $(2^4)^3 = 2^{12}$. $2^4 2^3 = 2^7$. $2^3 2^4 = 2^7$.
- This is a quadratic for $\frac{1}{x}$ with solutions $\frac{1}{x} = -1$ or $\frac{1}{x} = -3$. So $x = -1$ or $x = -\frac{1}{3}$. Alternatively, we could multiply both sides by x^2 and solve the quadratic that we get.
- This is $\log_{10} 12$.
- The quadratic inside the brackets factorises, and this is $\log_3(x+2) + \log_3(x+1)$. Other answers are possible, such as $\log_3(2(x+2)) + \log_3((x+1)/2)$.
- The left-hand side is just 2 so we want $2 = x^3$. So $x = \sqrt[3]{2}$.
- The left-hand side is $1 + \log_x 2$ so we want $\log_x 2 = 2$. So $x^2 = 2$ and $x = \sqrt{2}$ (not $-\sqrt{2}$ because we're told that $x > 0$).
- Take $(x+5)$ to the power of each side to get $6x+22 = (x+5)^2$. Expand the square and rearrange for $x^2 + 4x + 3 = 0$. The solutions are $x = -1$ or $x = -3$. Check these solutions; $\log_4(16) = 2$ and $\log_2(4) = 2$.
- $\ln 1024 = \ln(2^{10}) = 10 \ln 2 = 10a$.
 - $\ln 40 = \ln 8 + \ln 5 = 3a + b$.
 - $\ln \sqrt{2/5} = \frac{1}{2} \ln 2/5 = \frac{1}{2}(a - b)$.
 - $\ln(1/10) = -\ln 10 = -\ln 2 - \ln 5 = -a - b$.
 - $\ln 1.024 = \ln 1024 + \ln 1/1000 = 10a + 3(-a - b) = 7a - 3b$.
 - There are other solutions, partly because $b = a \times \log_2 5$.
- $e^{x+y} + e^{y-x} - e^{x-y} - e^{-x-y} + e^{x+y} - e^{y-x} + e^{x-y} - e^{-x-y}$. That's $2e^{x+y} - 2e^{-x-y}$.
 $e^{x+y} + e^{y-x} + e^{x-y} + e^{-x-y} + e^{x+y} - e^{y-x} - e^{x-y} + e^{-x-y}$. That's $2e^{x+y} + 2e^{-x-y}$.
- $2^x = 3$ is what it means for x to be $\log_2 3$.
If $0.5^x = 3$ then $2^{-x} = 3$ so $x = -\log_2 3$. Alternatively, just write down $x = \log_{0.5} 3$.
If $4^x = 3$ then $2^{2x} = 3$ so $x = \frac{1}{2} \log_2 3$. Alternatively, just write down $x = \log_4 3$.
- $1^x = 1$ is true for all real x . 1^x is never equal to 3 for real x .
- $0^b = 0$ for any real $b > 0$. a^0 is never 0.
- We always have $a^0 = 1$ for $a > 0$. So $\log_a 1 = 0$. Note that a^x is a strictly increasing function if $a > 1$ and strictly decreasing if $a < 1$, so if $a^x = 1$ then x must be 0.
- We have $\log_{10} x = 10^6$, which is a million. So x is ten to the power of a million. That's got a million zeros at the end.

15. We have $\log_a \left(\frac{x}{y} \right) = \log_a (xy^{-1}) = \log_a x + \log_a y^{-1}$ using the product rule, and then $\log_a y^{-1} = -\log_a y$ using the power rule.

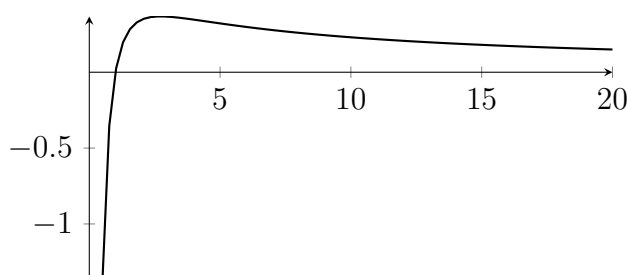
16. Multiply both sides by e^x and rearrange to get $e^{2x} - 4e^x + 1 = 0$. This is a quadratic for e^x . Solve it for $e^x = 2 \pm \sqrt{3}$. So $x = \ln(2 \pm \sqrt{3})$.

Following the previous working, we can see that we'll get two roots for e^x if $c^2 - 4 > 0$. But we need these to be positive roots, so we need $c > 2$. If $c = 2$ there's a repeated root. If $c < 2$ there are no roots.

17. Move both terms onto the left-hand side and use the fact that $(N + \sqrt{N^2 - 1})(N - \sqrt{N^2 - 1}) = N^2 - (N^2 - 1) = 1$; that's the difference of two squares. Remember that $\ln 1 = 0$.

As a result, our solutions to the previous question, $\ln(2 \pm \sqrt{3})$, are actually $\pm \ln(2 + \sqrt{3})$, revealing a lovely symmetry. But you could have spotted that from the equation, of course!

18. We have $\ln(x^y) = \ln(y^x)$ which we can simplify down to $y \ln x = x \ln y$. Now rearrange to get $\frac{\ln x}{x} = \frac{\ln y}{y}$. You might choose to put this fraction the other way up, or to square both sides or something, so your $f(x)$ might not be the same as mine. Here's a sketch of $y = \frac{\ln(x)}{x}$.



19. This is $(a^{\log_a b})^k = b^k$.

We can use a similar bit of algebra with $k = \log_b c$.

$$a^x = a^{\log_a b \log_b c} = (a^{\log_a b})^{\log_b c} = (b)^{\log_b c} = c.$$

So $a^x = c$ and therefore $x = \log_a c$.

If we relabel the previous result, we can write $(\log_c a)(\log_a b) = \log_c b$. Now divide by $\log_c a$ to get

$$\log_a b = \frac{\log_c b}{\log_c a}$$

In particular, if we take c to be the number e , then we can write that fraction with \ln instead of \log_e . This is handy because it shows that we can always write logarithms like $\log_a x$ in terms of \ln . All other $\log_a x$ graphs are just simple transformations of the $\ln x$ graph.

MAT Questions**MAT 2014 Q1B**

- The function $y = x^2 - 2x + 2 = (x - 1)^2 + 1$ if we complete the square.
- The minimum occurs when $x = 1$ and the value at that minimum is $y = 1$.
- Consider now the function $y = \log_{10}(f(x))$ where $f(x)$ is the quadratic above. This graph has a local minimum at $x = 1$, and the value there will be $\log_{10} 1 = 0$. Only one of the graphs looks like that.
- The answer is (e).

MAT 2015 Q1H

- We can start by using the fact that if $\log_b a = c$ then $a = b^c$.
- Here $(4 - 5x^2 - 6x^3) = (x^2 + 2)^2$.
- That polynomial simplifies to $x^4 + 6x^3 + 9x^2 = 0$.
- There's a repeated root at $x = 0$, and the polynomial factorises as $x^2(x + 3)^2 = 0$.
- We should check our solutions; $\log_2(4) = 2$ and $\log_{11}(4 - 5 \times 9 + 6 \times 27) = \log_{11}(121) = 2$.
- So there are two distinct solutions.
- The answer is (c).

MAT 2017 Q1I

- The equation simplifies to

$$x^2 + x \log_a \left(\frac{c}{b} \right) + \log_a \left(\frac{1}{b} \right) \log_a(c) = 0.$$

- This quadratic factorises as

$$(x - \log_a(b))(x + \log_a(c)) = 0,$$

where I've used the law of logarithms that $\log_a c - \log_a b = \log_a \left(\frac{c}{b} \right)$.

- This quadratic has two distinct roots, unless $-\log_a b = \log_a c$. That happens precisely when $c = \frac{1}{b}$.
- The answer is (d).

MAT 2008 Q1B

Let $x = \log_{10} \pi$. Note that $0 < x < 1$ because $1 < \pi < 10$. The four values are

$$x, \quad \sqrt{2x}, \quad x^{-3} \quad \frac{2}{x}.$$

Now we want to compare these terms. We have $x < \sqrt{2x}$ if $x^2 < 2x$ which happens if $x < 2$. Also $x < x^{-3}$ because $x < 1$. Also $x < 2/x$ because $x < \sqrt{2}$.

The answer is (a).

MAT 2008 Q1E

Ignore small powers of x inside each pair of round brackets. We get something like

$$\left\{ \left[(2x^6 + \dots)^3 + (3x^8 + \dots)^4 \right]^5 + \left[(3x^5 + \dots)^5 + (x^7 + \dots)^4 \right]^6 \right\}^3$$

Now apply the powers on the round brackets and compare terms again

$$\left\{ \left[2^3 x^{18} + 3^4 x^{32} + \dots \right]^5 + \left[3^5 x^{25} + x^{28} + \dots \right]^6 \right\}^3$$

Take the largest power inside each square bracket and apply the power on that bracket

$$\left\{ 3^{20} x^{160} + x^{168} + \dots \right\}^3$$

Take the largest power inside the curly brackets and apply the power on that bracket

$$x^{504} + \dots$$

The answer is (d).

MAT 2010 Q1E

First note that $\log_4 8 = \frac{3}{2}$ because $4 = 2^2$ and $8 = 2^3$, so $4^{3/2} = 8$.

Is $\log_2 3 > \frac{3}{2}$? Only if $3 > 2^{3/2}$ so only if $9 > 8$. Yes!

Is $\log_3 2 > \frac{3}{2}$? No, it's less than one.

Is $\log_5 10 > \frac{3}{2}$? Only if $10 > 5^{3/2}$, so only if $100 > 125$. No!

So only $\log_2 3$ is larger than $\frac{3}{2}$.

The answer is (a).

MAT 2012 Q1C

Simplify $(\sqrt{3})^3 = 3\sqrt{3}$.

Simplify $\log_3(9^2) = 4$.

Simplify $(3 \sin 60^\circ)^2 = (3\sqrt{3}/2)^2 = \frac{27}{4}$.

Simplify $\log_2(\log_2 8^5) = \log_2(\log_2 2^{15}) = \log_2(15)$.

The next thing I notice is that $15 < 16$ so $\log_2(15) < 4$. Aha, that means that it's less than $\log_3(9^2)$.

Then perhaps I could notice that $\frac{27}{4} > 4$. So my remaining candidates for the smallest of the numbers are $3\sqrt{3}$ and $\log_2(15)$. But $3\sqrt{3} = \sqrt{27} > 4$. So option (d) is the only one that's less than 4.

The answer is (d).

Extension

- We've seen above that for $0 < \alpha < 1$, the smallest is (a). For $\alpha > 1$, it turns out that α^{-3} is the smallest.
- $(8!)^9 = (8!)^8 \times 8!$ and $(9!)^8 = (8!)^8 \times 9^8$. Now 9^8 is clearly larger than $8!$ (each is the product of eight things, and in the case of 9^8 , each of those eight things is larger!). So $(9!)^8 > (8!)^9$.