Integral Transforms

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SYLLABUS

Motivation for a "function" with the properties of the Dirac δ-function. Test functions. Continuous functions are determined by $\phi \to \int f\phi$. Distributions and δ as a distribution. Differentiating distributions. (3 lectures)

Theory of Fourier and Laplace transforms, inversion, convolution. Inversion of some standard Fourier and Laplace transforms via contour integration.

Use of Fourier and Laplace transforms in solving ordinary differential equations, with some examples including δ.

Use of Fourier and Laplace transforms in solving partial differential equations; in particular, use of Fourier transform in solving Laplace's equation and the Heat equation. (5 lectures).

SUGGESTED READING

Sam Howison, *Practical Applied Mathematics* (CUP, 2005), chapters 9 & 10 (for distributions).
RI Richards & HK Youn, *The Theory of Distributions: A Nontechnical Introduction* (CUP).
PJ Collins, *Differential and Integral Equations* (OUP, 2006), Chapter 14
WE Boyce & RC DiPrima, *Elementary Differential Equations and Boundary Value Problems* (7th edition, Wiley, 2000). Chapter 6
KF Riley & MP Hobson, *Essential Mathematical Methods for the Physical Sciences* (CUP 2011) Chapter 5
HA Priestley, *Introduction to Complex Analysis* (2nd edition, OUP, 2003) Chapters 21 and 22

LECTURE LAYOUT

1. Motivation. Functions as distributions.
2-3. Test Functions. Distributions and differentiating distributions.
3-4 Laplace Transform. Properties.
4-5. Applications to ODEs
6. Convolution and Inversion.
7. Fourier Transform and applications.
8. Applications to PDEs

Remark 1 *Before we even get started we need to recognize that we don't really have a rigorous enough or general enough theory of integration on the real line. The Riemann integral (constructed in Prelims Analysis III) applies to bounded functions on a bounded interval. A more general theory – Lebesgue Integration – does exist and anyone interested can study this in further detail in the A4 Integration option.*

Somewhat simplistically there are two ways in which a function can fail to be integrable. A function can be so pathological that there is simply no hope for integrating it. However the Lebesgue integral is so very general that almost any function we might conceive of is ***Lebesgue measurable*** *and so in practice this issue does not arise.*

Rather we more typically meet functions that fail to be integrable because the area that integral would represent is simply infinite. Such examples are

$$\frac{\mathbb{1}_{(0,1)}(x)}{x}, \qquad \mathbb{1}_{(0,\infty)}(x), \qquad e^x \times \mathbb{1}_{(0,\infty)}(x).$$

There are some more subtle examples such as

$$\frac{\sin x}{x} \times \mathbb{1}_{(0,\infty)}(x),$$

where the absolute value of the function isn't integrable even though the integral is conditionally convergent (if you integrate to infinity via intervals of length 2π, cancellation of alternate positive and negative terms in each of these gives a finite result).

Especially when we come to the Laplace Transform one of the main properties of the Lebesgue integral that we shall be using is:

- *If f is a (measurable) function, g an integrable function and $|f| \leqslant g$ then f is integrable.*

One might think of this as a comparison test for integrals.

- *(**Riemann-Lebesgue Lemma.**) If f is an integrable function then*

$$\int_{-\infty}^{\infty} f(x)\cos ax \,\mathrm{d}x \to 0 \qquad \textit{and} \qquad \int_{-\infty}^{\infty} f(x)\sin ax \,\mathrm{d}x \to 0 \qquad \textit{as } a \to \infty.$$

(This is another cancellation-of-oscillations result, now because the oscillations get more and more rapid as $a \to \infty$. A consequence of this lemma is that Fourier coefficients a_n and b_n tend to zero as $n \to \infty$.)

Finally, when it comes to the discussion of distributions, the following definition will also be important.

Definition 2 *$f : \mathbb{R} \to \mathbb{R}$ is* ***locally integrable*** *if it is integrable on any bounded interval. So, for example, x^2 is not integrable on $\mathbb{R}$ but it is locally integrable.*

Chapter 1

The Dirac δ-function and Distributions

1.1 Motivation

There are many instances in mathematics where one might want to model a problem which – in the traditional sense – cannot be described by a function and will lead to singularities. Typical examples are:

- A point mass.
- A point heat source.
- An instantaneous impulse.

Example 3 ***(A Point Heat Source)*** *Consider the time-independent heat equation*[1] *in a bar* $-1 \leqslant x \leqslant 1$*:*

$$-T''(x) = g(x), \quad -1 < x < 1, \qquad T(-1) = 0 = T(1).$$

The function $g(x)$ *describes the heat being introduced or removed from the bar. What function* δ *would model a unit point source at* $x = 0$*? Necessarily we would need the following:*

$$\delta(x) = 0 \quad \textit{for} \quad x \neq 0; \qquad \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1.$$

[1] Why is there a minus sign on the left? The signs are chosen so that, for the time-dependent heat equation $\partial u/\partial t - \partial^2 u/\partial x^2 = g$, a positive source g (heat input) results in an increasing temperature.

Our immediate problem is that no function g, in the classical sense of the word, has these properties.

On the other hand we might nonetheless try to solve the the BVP above for such δ. As $T''(x) = 0$ for $x \neq 0$ and with the given boundary conditions we know

$$T(x) = \begin{cases} A(x-1) & x > 0; \\ B(1+x) & x < 0. \end{cases}$$

Physically we would also expect T to be continuous at $x = 0$ which means $-A = B$. Finally we would also need

$$1 = \int_{-1}^{1} -T''(x)\,\mathrm{d}x = [-T'(x)]_{-1}^{1} = -A + B,$$

so that $A = -1/2$, $B = 1/2$. So our solution is

$$T(x) = \begin{cases} \frac{1}{2}(1-x) & x > 0; \\ \frac{1}{2}(1+x) & x < 0. \end{cases}$$

And in some sense the function we are interested in is $\delta(x) = -T''(x)$. This doesn't help that much again as in the classical sense T is not even differentiable let alone twice-differentiable.

Example 4 ***(A Point Mass)*** *In a similar fashion, by Poisson's equation, a point mass M at the origin of the real line will generate a gravitational field $f(x)$ that satisfies*

$$f'(x) = -4\pi G\rho = 0 \quad \textit{for} \quad x \neq 0$$

but by Gauss's Flux Theorem we also have

$$\int_{-\infty}^{\infty} f'(x)\,\mathrm{d}x = -4\pi GM.$$

It would seem that if we could find an appropriate function $\delta(x)$ for the first example then $f'(x) = -4\pi GM\delta(x)$ would work here.

Whilst we are here might also notice from Gauss' Flux Theorem that we would expect

$$\int_{-\infty}^{a} f'(x)\,\mathrm{d}x = \begin{cases} -4\pi GM & a > 0; \\ 0 & a < 0; \end{cases} \qquad \Longrightarrow \qquad \int_{-\infty}^{a} \delta(x)\,\mathrm{d}x = \begin{cases} 1 & a > 0; \\ 0 & a < 0. \end{cases}$$

Example 5 ***(Kick Start)*** *Let $T > 0$ and consider the ODE*

$$m\ddot{x}(t) + kx(t) = I\,\delta(t-T), \qquad x(0) = \dot{x}(0) = 0.$$

This is the equation governing the extension of a mass m on a spring with spring constant k. The system remains at rest until a time T when an instantaneous impulse I is applied. Determine the extenstion $x(t)$.

Solution. The solution to the problem is of the form

$$x(t) = \begin{cases} A\cos\omega t + B\sin\omega t & t < T; \\ C\cos\omega t + D\sin\omega t & t > T; \end{cases}$$

wher $\omega^2 = k/m$. From the initial conditions we have that $A = B = 0$ and the system sits at rest for $t < T$. By continuity we also have that

$$C\cos\omega T + D\sin\omega T = 0.$$

However there is a discontinuity I in the momentum $m\dot{x}$ of the particle at $t = T$. So

$$m\omega(-C\sin\omega T + D\cos\omega T) = I.$$

Hence

$$\begin{pmatrix} \cos\omega T & \sin\omega T \\ -\sin\omega T & \cos\omega T \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{I}{m\omega} \end{pmatrix} \implies \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} \cos\omega T & -\sin\omega T \\ \sin\omega T & \cos\omega T \end{pmatrix} \begin{pmatrix} 0 \\ \frac{I}{m\omega} \end{pmatrix}$$

and

$$x(t) = \frac{I}{m\omega}\left\{-\sin\omega T\cos\omega t + \cos\omega T\sin\omega t\right\} = \frac{I\sin\omega(t-T)}{m\omega} \qquad \text{for } t \geqslant T.$$

■

1.2 Delta function

So how do we go about rigorously defining a "function" δ with the properties

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0; \qquad \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1.$$

We know that there is no such function in the classical sense, so we need to think how else we might convey information about functions and, with luck, find a more general setting in which δ might exist.

Suppose now that $\phi : \mathbb{R} \to \mathbb{R}$ is continuous. Then for any $\varepsilon > 0$ there exists $\Delta > 0$ such that

$$-\Delta < x < \Delta \quad \implies \quad -\varepsilon < \phi(x) - \phi(0) < \varepsilon.$$

Note that

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)\,\mathrm{d}x = \int_{-\infty}^{-\Delta} \delta(x)\phi(x)\,\mathrm{d}x + \int_{-\Delta}^{\Delta} \delta(x)\phi(x)\,\mathrm{d}x + \int_{\Delta}^{\infty} \delta(x)\phi(x)\,\mathrm{d}x = \int_{-\Delta}^{\Delta} \delta(x)\phi(x)\,\mathrm{d}x,$$

and so we would expect

$$\phi(0)-\varepsilon=(\phi(0)-\varepsilon)\int_{-\Delta}^{\Delta}\delta(x)\,\mathrm{d}x\leqslant\int_{-\Delta}^{\Delta}\delta(x)\phi(x)\,\mathrm{d}x\leqslant(\phi(0)+\varepsilon)\int_{-\Delta}^{\Delta}\delta(x)\,\mathrm{d}x=\phi(0)+\varepsilon.$$

As this is true for all ε then we have

$$\int_{-\infty}^{\infty}\delta(x)\phi(x)\,\mathrm{d}x=\phi(0)\qquad\text{when }\phi\text{ is continuous.}\tag{1.1}$$

This is quite a big steer towards the idea of a *generalized function* or *distribution.* As we will see a continuous function $f(x)$ can be reconstructed from knowledge of integrals such as

$$\langle f,\phi\rangle:=\int_{-\infty}^{\infty}f(x)\phi(x)\,\mathrm{d}x$$

where $\phi(x)$ is any continuous function. Piecewise continuous functions could also largely be reconstructed, but we would be uncertain quite what values were taken at the discontinuities.

We see from (1.1) that our desired function $\delta(x)$ could fit within this framework of generalized functions. Moreover if we are careful with our choice of functions $\phi(x)$ we will see that it is possible to do calculus with these generalized functions.

1.3 Test Functions and Distributions

So it seems that to understand δ we need to set aside our narrow view of a function f being defined simply at points and instead try to understand the map

$$\phi\mapsto\langle f,\phi\rangle=\int_{-\infty}^{\infty}f(x)\phi(x)\,\mathrm{d}x\tag{1.2}$$

where $\phi(x)$ is a continuous function. Certainly continuous functions f can be uniquely represented this way and we have seen that δ can be understood this way as "evaluation at 0". Note though that this effectively treats $f(x)$ as a functional on the space of continous functions.

However, we might be more ambitious and hope to differentiate these generalized functions. Given a generalized function f, can we make sense of its derivative f' as another generalized function? This would be

$$\phi\mapsto\langle f',\phi\rangle=\int_{-\infty}^{\infty}f'(x)\phi(x)\,\mathrm{d}x.$$

What sense might this integral on the RHS make? *If* we could apply integration-by-parts and *if* ϕ were differentiable then we'd have

$$\int_{-\infty}^{\infty} f'(x)\phi(x)\,\mathrm{d}x = [f(x)\phi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\phi'(x)\,\mathrm{d}x.$$

The second integral makes sense as this is just the generalized function f evaluated on ϕ'. And not unreasonably we might also require that

$$\lim_{x\to\infty}\phi(x) = \lim_{x\to-\infty}\phi(x) = 0.$$

This would mean that f' was the generalized function

$$\phi \mapsto -\int_{-\infty}^{\infty} f(x)\phi'(x)\,\mathrm{d}x. \tag{1.3}$$

So we are going to have to shift the goal posts a little now. Instead of considering continuous functions $\phi(x)$ generally, if we want generalized functions to be differentiable and for the derivative to be itself a generalized function, then the functions $\phi(x)$ need to be infinitely differentiable On the basis of the calculation above it seems we would also like $\phi(\infty) = \phi(-\infty) = 0$.

Thus we make the following definition.

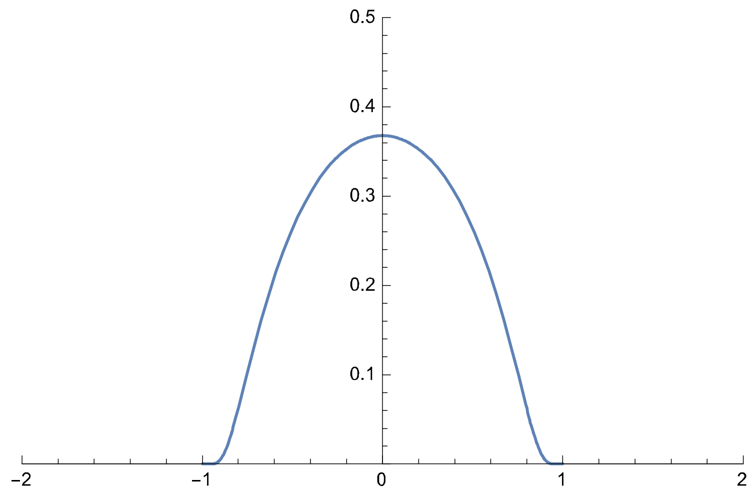
Definition 6 *A map $\phi : \mathbb{R} \to \mathbb{R}$ is a **test function** if it is smooth (i.e. infinitely differentiable) and if there exists X such that $\phi(x) = 0$ when $|x| > X$.*

- It is an easy check to show that the test functions form a vector space $\mathcal{D}$.
- We have restricted our attention from continuous functions to test functions so that we can differentiate generalized functions. We shall see though that test functions are still plentiful enough that a continous function can be reconstructed from (1.2).

Example 7 *The function*

$$\phi(x) = \begin{cases} \exp\left(\frac{1}{x^2-1}\right) & |x| < 1; \\ 0 & |x| \geqslant 1; \end{cases}$$

is a test function.



Solution. It is clearly infinitely differentiable for $x \neq \pm 1$. We also have

$$\begin{aligned}\lim_{x \nearrow 1} \frac{\phi(x)-\phi(1)}{x-1} &= \lim_{x \nearrow 1} \frac{1}{x-1} \exp\left(\frac{1}{x^2-1}\right) \\ &= \lim_{x \nearrow 1} \frac{1}{x-1} \exp\left(\frac{1/2}{x-1}\right) \exp\left(-\frac{1/2}{x+1}\right) \\ &= \exp\left(-\frac{1}{4}\right) \lim_{x \nearrow 1} \frac{1}{x-1} \exp\left(\frac{1/2}{x-1}\right) \\ &= \exp\left(-\frac{1}{4}\right) \lim_{t \to -\infty} t \exp\left(\frac{t}{2}\right) = 0.\end{aligned}$$

So that $\phi'(1) = 0$ and as ϕ is even then $\phi'(-1) = 0$.

For $x \neq \pm 1$ we have

$$\phi'(x) = \frac{-2x}{(x^2-1)^2} \exp\left(\frac{1}{x^2-1}\right)$$

and for $k \geqslant 1$ might make the inductive hypothesis that, for $x \neq \pm 1$,

$$\phi^{(k)}(x) = \frac{p_k(x)}{(x^2-1)^{2k}} \exp\left(\frac{1}{x^2-1}\right),$$

where $p_k(x)$ is a polynomial of degree less than or equal to $3k$. If true at k then

$$\begin{aligned}
\phi^{(k+1)}(x) &= \left\{ \frac{p_k'(x)}{(x^2-1)^{2k}} - \frac{2xp_k(x)}{(x^2-1)^{2k+2}} - \frac{4kxp_k(x)}{(x^2-1)^{2k+1}} \right\} \exp\left(\frac{1}{x^2-1}\right) \\
&= \left\{ \frac{p_k'(x)(x^2-1)^2 - 2xp_k(x) - 4kx(x^2-1)p_k(x)}{(x^2-1)^{2k+2}} \right\} \exp\left(\frac{1}{x^2-1}\right) \\
&= \frac{p_{k+1}(x)}{(x^2-1)^{2k+2}} \exp\left(\frac{1}{x^2-1}\right).
\end{aligned}$$

So our hypothesis holds for all k and by an argument similar to the one showing $\phi'(1) = 0$ we see that $\phi^{(k)}(1) = 0$ for all $k \geqslant 1$.

Hence $\phi(x)$ is a test function. ■

A similar calculation would show more generally that

$$\phi(x) = \begin{cases} \exp\left(\frac{C}{(x-a)(x-b)}\right) & a < x < b \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

is a test function. In the end, however, we don't really care too much about the test functions, as long as we know they exist and there are 'enough' of them to let us work with distributions.

It remains an important point that continuous functions can be reconstructed from knowledge of the integrals in (1.2). More precisely we show the following.

Theorem 8 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x = 0$$

for all test functions ϕ. Then $f = 0$.

Proof. Suppose for a contradiction that $f(x_0) \neq 0$ for some x_0. Without loss of generality say $f(x_0) > 0$. If we set $\varepsilon = f(x_0)/2 > 0$ then by continuity we can find $\Delta > 0$ such that $f(x_0) > \varepsilon$ for $x_0 - \Delta < x < x_0 + \Delta$. If we take the test function from (1.4) with $a = x_0 - \Delta$ and $b = x_0 + \Delta$

$$\int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x = \int_{x_0-\Delta}^{x_0+\Delta} f(x)\phi(x)\,\mathrm{d}x > \varepsilon \int_{x_0-\Delta}^{x_0+\Delta} \phi(x)\,\mathrm{d}x > 0,$$

which is the required contradiction. ■

For each continuous function f we then have a functional F_f on the space $\mathcal{D}$ of test functions

$$F_f : \phi \mapsto F_f(\phi) = \langle f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x.$$

What the above theorem shows is that the map $f \mapsto F_f$ is 1–1.

If we wish to view the δ-function in the same way then, following (1.1), we should think of δ as the following functional

$$\delta : \phi \mapsto \phi(0).$$

Thus we are almost ready to define generalized functions/distributions.

Definition 9 *A* ***distribution*** *or* ***generalized function*** *F is a linear functional from $\mathcal{D}$ to $\mathbb{R}$ which is continuous in the following sense:*

- *F is continuous if whenever ϕ and ϕ_n ($n \geqslant 1$) are test functions which are all zero outside some bounded interval I and each $\phi_n^{(k)}$ converges uniformly to $\phi^{(k)}$ as $n \to \infty$, then $F(\phi_n) \to F(\phi)$.*

We write $\mathcal{D}'$ for the space of distributions. Also, we write $\langle F, \phi \rangle$ for the real number $F(\phi)$. When we want emphasise the range of the distribution (which really means the range of the test functions), we may write $F(x)$ instead of just F.

Remark 10 *Informally one might think of distributions as functions, which whilst not defined at points, do have a well-defined average on any neighbourhood of a point. This is consistent with the general point that the integral of a function is normally smoother than the function itself (whereas differentiation normally decreases smoothness — but not for test functions, of course!).*[2] *Thus, an average can make sense where a pointwise view does not.*

Remark 11 *The above requirement of continuity may seem somewhat technical but it is precisely what we want if we desire the derivative of a distribution to be a distribution.*

Remark 12 *It is a relatively easy check to show that $\mathcal{D}'$ is indeed a vector space. Note that $\mathcal{D}'$ isn't the algebraic dual of $\mathcal{D}$ — the space of all functionals — but rather a subspace of the algebraic dual.*

Proposition 13 *Given a locally integrable function f then F_f is a distribution. Such distributions are called* ***regular distributions.***

[2] For example, think of integrating and/or differentiation functions like x^n, or $\max(x, 0)$.

Proof. Clearly F_f is linear. Suppose, for each k, that $\phi_n^{(k)} \xrightarrow{\mathrm{u}} \phi^{(k)}$ and that the ϕ_n, ϕ are zero outside some bounded interval I. Then

$$\begin{aligned} F_f(\phi_n) = \langle f, \phi_n \rangle &= \int_{-\infty}^{\infty} f(x)\phi_n(x)\,\mathrm{d}x \\ &= \int_I f(x)\phi_n(x)\,\mathrm{d}x \\ &\to \int_I f(x)\phi(x)\,\mathrm{d}x \qquad \text{[by uniform convergence]} \\ &= \langle f, \phi \rangle \\ &= F_f(\phi). \end{aligned}$$

■

- Note, though, that different functions may represent the same distribution.

Example 14 *The functions*

$$H_1(x) = \begin{cases} 1 & x > 0; \\ 0 & x \leqslant 0; \end{cases} \qquad H_2(x) = \begin{cases} 1 & x \geqslant 0; \\ 0 & x < 0; \end{cases}$$

both lead to the same distribution

$$H : \phi \to \langle H, \phi \rangle = \int_0^{\infty} \phi(x)\,\mathrm{d}x.$$

This distribution is called the ***Heaviside function****.*

- But different continuous functions induce different distributions (as a consequence of Theorem 8).

Proposition 15 *δ is a distribution.*

- No locally integrable function f induces δ. This means δ is a **singular distribution.**

Proof. Clearly δ is linear. Suppose, for each k, that $\phi_n^{(k)} \xrightarrow{\mathrm{u}} \phi^{(k)}$. Then in particular (with $k = 0$) we have $\phi_n \to \phi$ pointwise. So $\phi_n(0) \to \phi(0)$. ■

Example 16 *(**Approximating** $\delta(x)$) Consider the sequence of functions*

$$\delta_n(x) = \begin{cases} \frac{n}{2} & |x| < \frac{1}{n}; \\ 0 & \text{otherwise}; \end{cases}$$

We say that a sequence of distributions (F_n) converges to a distribution F if $\langle F_n, \phi\rangle \to \langle F, \phi\rangle$ for all test functions ϕ. If ϕ is a test function then in particular it is continuous at 0. By the MVT for integrals, we have for some $\xi_n \in (-1/n, 1/n)$

$$\langle \delta_n, \phi\rangle = \int_{-\infty}^{\infty} \delta_n(x)\phi(x)\,\mathrm{d}x = \frac{n}{2}\int_{-1/n}^{1/n} \phi(x)\,\mathrm{d}x = \frac{n}{2}\phi(\xi_n)\int_{-1/n}^{1/n} \mathrm{d}x = \phi(\xi_n) \to \phi(0),$$

by continuity. Hence $\langle \delta_n, \phi\rangle \to \langle \delta, \phi\rangle$ as $n \to \infty$.

Proposition 17 *Suppose that the integrable function $f(x)$ is continuous at $x = 0$. Then, as $n \to \infty$,*

$$\langle \delta_n, f\rangle = \int_{-\infty}^{\infty} \delta_n(x) f(x)\,\mathrm{d}x \to f(0)$$

and hence $\delta_n \to \delta$.

Proof. Simply replace ϕ with f in Example 16. ■

Remark 18 *This proposition shows us that, although it is defined by its action on a test function, the delta function also works when integrated against a continuous function. One can define δ and other distributions via limits of approximating sequences, but this approach is fraught with technical problems (for example, would two sequences, both approximating δ, give the same result for δ' (defined below)?).*

1.4 More properties of distributions

We have already noted that the distributions form a vector space $\mathcal{D}'$. We now give them some more important properties; in each case, we do so in a way that consistently extends the 'ordinary' properties of regular distributions.

Definition 19 *(Translation of a distribution). Let $F(x)$ be a distribution and $a \in \mathbb{R}$. The translation of F through a, written $F(x-a)$, is defined by its action*

$$\langle F(x-a), \phi(x)\rangle = \langle F(x), \phi(x+a)\rangle.$$

When f is a regular distribution corresponding to an integrable function f, this is just a change of variable in the integral:

$$\langle f(x-a), \phi(x)\rangle = \int_{-\infty}^{\infty} f(x-a)\phi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(u)\phi(u+a)\,\mathrm{d}u = \langle f(x), \phi(x+a)\rangle.$$

When the distribution is the delta function, this is known as the *sifting property*:

$$\langle \delta(x-a), \phi(x) \rangle = \phi(a).$$

And by a very simple extension of Proposition 17, this applies to any locally integrable function which is continous at a:

$$\langle \delta(x-a), f(x) \rangle = f(a).$$

Remark 20 *It's essentially this property that earned the distribution its δ notation. Compare this with the discrete version*

$$\sum_j \delta_{ij} a_{jk} = a_{ik}$$

(here δ_{ij} is the Kronecker delta). One might view $\delta(x-a)$ as a continuous version of δ_{ij}.

Remark 21 *The delta function is only 'active' where its argument vanishes (e.g., for $\delta(x-a)$ this is at $x = a$); if the support of a test function ϕ does not contain $x = a$, then $\langle \delta_a, \phi \rangle = 0$. Thus, the delta function is 'localised' at $x = a$. In many uses of the delta function, for example in solving differential equations, we work on an interval rather than all of $\mathbb{R}$. Because our test functions were defined on all of $\mathbb{R}$, for the utmost rigour (mortis) we should redefine a new class of test functions adapted to our interval. However, the localised nature of the delta function makes this unnecessary, and it is perfectly safe to go ahead and just use δ. That is what it was dreamed up for!*

Definition 22 *If f is a smooth function (i.e. it has derivatives of all orders) and F is a distribution then the distribution fF has action $\langle fF, \phi \rangle = \langle F, f\phi \rangle$.*

For a regular distribution F and a test function ϕ we have

$$\langle F, \phi \rangle = \int_{-\infty}^{\infty} F(x)\phi(x)\,\mathrm{d}x$$

and so

$$\begin{aligned}\langle fF, \phi \rangle &= \int_{-\infty}^{\infty} f(x)F(x)\phi(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} F(x)f(x)\phi(x)\,\mathrm{d}x = \langle F, f\phi \rangle\end{aligned}$$

(that is, $f\phi$ is a test function). Thus, this definition (for regular and singular distributions) is consistent with the usual interpretation for a regular distribution.

We should show that fF satisfies the properties of a distribution (linearity and continuity). Clearly fF is linear. Suppose that $\phi_n \stackrel{\mathrm{u}}{\to} \phi$ and that the ϕ_n, ϕ are zero outside some bounded interval I. Then $(f\phi_n) \stackrel{\mathrm{u}}{\to} (f\phi)$ and hence

$$\langle fF, \phi_n \rangle = \langle F, f\phi_n \rangle \to \langle F, f\phi \rangle = \langle fF, \phi \rangle.$$

Hence fF is continuous and so a distribution.

Surprisingly, perhaps, it is also possible to differentiate distributions. We already considered the possibility at (1.3) and so we define:

Definition 23 *Given a a distribution F and test function ϕ we define*

$$\langle F', \phi \rangle = -\langle F, \phi' \rangle.$$

- Note that this agrees with normal differentiation for regular distributions from differentiable functions: say that f is differentiable, so that

$$\langle f', \phi \rangle = \int_{-\infty}^{\infty} f'(x)\phi(x)\,\mathrm{d}x = [f(x)\phi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\phi'(x)\,\mathrm{d}x = -\langle f, \phi' \rangle.$$

- Also if F is a distribution and f a smooth function then we have the product rule

$$(fF)' = f'F + fF'$$

as for any test function ϕ we have

$$\begin{aligned} \langle (fF)', \phi \rangle &= -\langle fF, \phi' \rangle \\ &= -\langle F, f\phi' \rangle \\ &= -\langle F, (f\phi)' \rangle + \langle F, f'\phi \rangle \\ &= \langle F', f\phi \rangle + \langle f'F, \phi \rangle \\ &= \langle fF', \phi \rangle + \langle f'F, \phi \rangle. \end{aligned}$$

Proposition 24 *If F is a distribution then so is F'.*

Proof. Clearly F' is linear. Further if $\phi_n^{(k)} \stackrel{\mathrm{u}}{\to} \phi^{(k)}$ and ϕ_n, ϕ are zero outside the bounded interval I then in particular $(\phi_n')^{(k)} \stackrel{\mathrm{u}}{\to} (\phi')^{(k)}$ and hence

$$\begin{aligned} \langle F', \phi_n \rangle &= -\langle F, \phi_n' \rangle \\ &\to -\langle F, \phi' \rangle \qquad \text{[by the continuity of } F\text{]} \\ &= \langle F', \phi \rangle \qquad \text{[by definition of } F'\text{]}. \end{aligned}$$

So F' is continuous and is a distribution. ■

Remark 25 *This proof shows that distributions inherit the infinite differentiability of test functions. If we had defined test functions to have only a finite number of derivatives, then the same would have applied to the corresponding distributions. Such a theory, although possible, would be unrewardingly cumbersome.*

We can now do a remarkable thing: we can differentiate a function with a singularity such as a jump discontinuity at a point (the Heaviside function is a simple example) and interpret the result without having to take limits from the left and right. Such functions fit naturally into the framework of distributions. We can go further and differentiate singular distributions such as δ.

Example 26

$$\text{(a)} \qquad H' = \delta, \qquad \text{(b)} \qquad \langle \delta', \phi \rangle = -\phi'(0).$$

Solution. (a) Recall that the Heaviside function satisfies

$$\langle H, \phi \rangle = \int_{-\infty}^{\infty} H(x)\phi(x)\,\mathrm{d}x = \int_0^{\infty} \phi(x)\,\mathrm{d}x.$$

So

$$\langle H', \phi \rangle = -\langle H, \phi' \rangle = -\int_0^{\infty} \phi'(x)\,\mathrm{d}x = \phi(0) = \langle \delta, \phi \rangle.$$

(b) We also have

$$\langle \delta', \phi \rangle = -\langle \delta, \phi' \rangle = -\phi'(0).$$

■

Remark 27 *If f has continuous derivative f', then $\langle \delta', f \rangle = -f'(0)$. Just as we can extend the action of δ from test functions to continuous functions, so we can extend the action of δ' to continuously differentiable functions.*

Example 28 *Find the first three derivatives of $f(x) = |x|$.*

Solution. As f is differentiable at $x \neq 0$ then $f'(x) = 1$ for $x > 0$ and $f'(x) = -1$ for $x < 0$. This is sufficient to define $f'(x)$ as a distribution and we in fact see

$$f'(x) = 2H(x) - 1.$$

Note that we have not determined the gradient of f at 0; this is unnecessary to define $f'(x)$ as a distribution. By the previous example we then have

$$f''(x) = 2\delta(x) \qquad \text{and} \qquad f'''(x) = 2\delta'(x).$$

■

Example 29 *The distributional derivative can be found by the ordinary calculus method:*

$$\delta'(x) = \lim_{h\to 0} \frac{\delta(x+h) - \delta(x)}{h}.$$

Solution. We have (expanding our notation to show the arguments of δ and ϕ)

$$\begin{aligned} \left\langle \frac{\delta(x+h) - \delta(x)}{h}, \phi(x) \right\rangle &= \frac{\phi(-h) - \phi(0)}{h} \\ &\to -\phi'(0) \quad \text{as } h \to 0 \\ &= \langle \delta', \phi \rangle. \end{aligned}$$

The reader may like to show that $H' = \delta$ in the same way. ■

Proposition 30 *Every distribution F has an antiderivative G such that $G' = F$.*

Proof. Let ϕ_0 be a fixed test function with total integral 1. Given any test function ϕ we can write $\phi = K\phi_0 + \phi_1$ where K is the total integral of ϕ and ϕ_1 has total integral 0. The point of this is that

$$\psi(x) = \int_{-\infty}^{x} \phi_1(x)\,\mathrm{d}x$$

is a test function and $\psi'(x) = \phi_1(x)$. We then define G by

$$\langle G, \phi \rangle = -\langle F, \psi \rangle.$$

Note that ϕ' has total integral 0 and so $(\phi')_1 = \phi'$ when ϕ' is decomposed as above; this means that the ψ corresponding to ϕ' is just ϕ. Hence we have

$$\langle G', \phi \rangle = -\langle G, \phi' \rangle = \langle F, \phi \rangle$$

and $G' = F$. ■

Example 31 *The product of two distributions need not be a distribution. This follows from the fact that the product fg of two locally integrable functions f and g need not be locally integrable. For example, consider*

$$f(x) = g(x) = \frac{1}{\sqrt{x}} \mathbb{1}_{(0,1)}(x).$$

Remark 32 ***(Some Historical Background)*** *Efforts to rigorously handle the mathematics behind point sources, point charges and point masses date back to Cauchy and Fourier. In the late nineteenth century Oliver Heaviside used Fourier series to model the unit impulse. The δ-function notation dates back to Paul Dirac's 1930 influential book "Principles of Quantum Mechanics". The French mathematician, Laurent Schwartz, developed the theory of distributions in the late 1940s to rigorously handle such notions, for which he was awarded the Fields medal in 1950.*

Chapter 2

Laplace Transform. Applications to ODEs.

This section is about the *Laplace transform* which is one of a number of important integral transforms in mathematics. An important aspect of the Laplace transform is that it can transform a differential equation into an algebraic one: ideally a differential equation in $f(x)$ is transformed into an algebraic one in its transform $\overline{f}(p)$ which we might solve with simple algebraic manipulation. Our remaining problem is then the inverse problem: to recognize this transform and so find the solution $f(x)$ to the original differential equation that transforms to this $\overline{f}(p)$.

Definition 33 *Let $f(x)$ be a real- or complex-valued function defined when $x > 0$. Then the **Laplace transform** $\overline{f}(p)$ of $f(x)$ is defined to be*

$$\overline{f}(p) = \int_0^\infty f(x)\mathrm{e}^{-px}\,\mathrm{d}x, \tag{2.1}$$

for those complex p where this integral exists. $\overline{f}$ is also commonly denoted as $\mathcal{L}f$ and the Laplace Transform itself as $\mathcal{L}$.

Remark 34 *The Laplace Transform of the probability density function of a positive random variable is essentially the moment generating funbction: $\overline{f}_X(p) = \mathbb{E}[\mathrm{e}^{-pX}] = M_X(-p)$.*

Example 35 *Let $f(x) = \mathrm{e}^{ax}$ where a is a complex number. Then*

$$\overline{f}(p) = \int_0^\infty \mathrm{e}^{ax}\mathrm{e}^{-px}\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-(p-a)x}\,\mathrm{d}x = \left[\frac{\mathrm{e}^{-(p-a)x}}{a-p}\right]_0^\infty = \frac{1}{p-a}$$

provided $\operatorname{Re} p > \operatorname{Re} a$. As $|\mathrm{e}^z| = \mathrm{e}^{\operatorname{Re} z}$, note that $\overline{f}(p)$ is undefined when $\operatorname{Re} p \leqslant \operatorname{Re} a$.

Example 36 *Let $f_n(x) = x^n$ where $n \geqslant 0$ is an integer. Then, provided $\mathrm{Re}\, p > 0$,*

$$\overline{f_n}(p) = \int_0^\infty x^n \mathrm{e}^{-px}\,\mathrm{d}x = \frac{-1}{p}\left[x^n \mathrm{e}^{-px}\right]_0^\infty + \frac{n}{p}\int_0^\infty x^{n-1}\mathrm{e}^{-px}\,\mathrm{d}x = \frac{n}{p}\overline{f_{n-1}}(p).$$

Now

$$\overline{f_0}(p) = \int_0^\infty \mathrm{e}^{-px}\,\mathrm{d}x = \frac{1}{p},$$

so that

$$\overline{f_n}(p) = \frac{n}{p} \times \overline{f_{n-1}}(p) = \frac{n}{p} \times \frac{n-1}{p} \times \cdots \times \frac{1}{p} \times \overline{f_0}(p) = \frac{n!}{p^{n+1}}.$$

Again the integral in the definition of $\overline{f_n}(p)$ is undefined when $\mathrm{Re}\, p \leqslant 0$.

Remark 37 *More generally the Laplace transform of x^a, where $a > -1$ is a real number, is $\Gamma(a+1)/p^{a+1}$ where $\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$ is the Gamma Function. See Sheet 1, Exercise 6.*

Example 38 *Let $f(x) = \cos ax$ and $g(x) = \sin ax$. Then*

$$\overline{f}(p) = \frac{p}{p^2 + a^2}; \qquad \overline{g}(p) = \frac{a}{p^2 + a^2}. \tag{2.2}$$

Solution. By integration by parts, and provided $\mathrm{Re}\, p > |\mathrm{Im} a|$,

$$\begin{aligned}
\overline{f}(p) &= \int_0^\infty \mathrm{e}^{-px}\cos ax\,\mathrm{d}x = \frac{-1}{p}\left[\mathrm{e}^{-px}\cos ax\right]_0^\infty - \frac{a}{p}\int_0^\infty \mathrm{e}^{-px}\sin ax\,\mathrm{d}x = \frac{1}{p} - \frac{a}{p}\overline{g}(s) \\
\overline{g}(p) &= \int_0^\infty \mathrm{e}^{-px}\sin ax\,\mathrm{d}x = \frac{-1}{p}\left[\mathrm{e}^{-px}\sin ax\right]_0^\infty + \frac{a}{p}\int_0^\infty \mathrm{e}^{-px}\cos ax\,\mathrm{d}x = \frac{a}{p}\overline{f}(p).
\end{aligned}$$

Solving the simultaneous equations

$$\overline{f}(p) + \frac{a}{p}\overline{g}(p) = \frac{1}{p}, \qquad \overline{g}(p) = \frac{a}{p}\overline{f}(p),$$

gives the expressions in (2.2).

An alternative and faster way to these expressions is the following. Let $h(x) = \mathrm{e}^{\mathrm{i}ax}$ where a is real. By Example 35 we have

$$\bar{h}(p) = \frac{1}{p - \mathrm{i}a} = \frac{p + \mathrm{i}a}{p^2 + a^2}.$$

Taking real parts gives $\overline{f}(p) = p/(p^2+a^2)$ and taking imaginary parts gives $\overline{g}(p) = a/(p^2+a^2)$. As these expressions hold for all real values of a then by the Identity Theorem they hold for all valid complex numbers a. ■

Example 39 *Let $a > 0$. The Laplace transform of $\delta(x-a)$ is e^{-ap}.* [1]

Proof. By the Sifting Property we have

$$\int_0^\infty \mathrm{e}^{-px}\delta(x-a)\,\mathrm{d}x = \mathrm{e}^{-ap}.$$

■

Example 40 *Let $a > 0$. The Laplace transform of*

$$H(x-a) = \begin{cases} 0 & 0 < x \leqslant a, \\ 1 & a < x \end{cases}$$

equals e^{-ap}/p.

Solution. We have

$$\int_0^\infty H(x-a)\mathrm{e}^{-px}\,\mathrm{d}x = \int_a^\infty \mathrm{e}^{-px}\,\mathrm{d}x = \left[\frac{\mathrm{e}^{-px}}{-p}\right]_a^\infty = \frac{\mathrm{e}^{-ap}}{p}.$$

■

In all the examples we have seen, we can note that if the Laplace transform $\overline{f}(p_0)$ exists (i.e. the relevant integral coverges) for a particular $p_0 \in \mathbb{C}$ then $\overline{f}(p)$ exists whenever $\operatorname{Re} p \geqslant \operatorname{Re} p_0$ which we formally state below.

Proposition 41 *Let $f(x)$ be a complex-valued function defined when $x > 0$, such that the integral (2.1) in the definition of $\overline{f}(p_0)$ exists for some complex number p_0. Then $\overline{f}(p)$ exists for all $\operatorname{Re} p \geqslant \operatorname{Re} p_0$.*

Proof. If $\operatorname{Re} p \geqslant \operatorname{Re} p_0$ then

$$\left|f(x)\mathrm{e}^{-px}\right| \leqslant \left|f(x)\mathrm{e}^{-p_0x}\right|$$

and hence by the comparison test for integrals (mentioned in the preamble to the notes) we see that $f(x)\mathrm{e}^{-px}$ is integrable on $(0,\infty)$. ■

We also note the following:

[1] It is usual to extend this to $a = 0$ by defining $\bar{\delta}(p) = 1$, which (for example by taking the Laplace Transform of $H'(x)$) amounts to choosing $H(0) = 1$; it says that the interval of interest includes the origin, and one can 'kick-start' a differential equation at $x = 0$. This is a rare example where the definition of a distribution at a point matters. Another example is the choice between $\mathbb{P}[X \leq x]$ and $\mathbb{P}[X < x]$ as the definition of the cumulative distribution function (the former is more common, and corresponds to $H(0) = 1$).

Proposition 42 *Let $f(x)$ be a continuous complex-valued function on $[0,\infty)$ such that $\overline{f}(p_0)$ exists. Then $\overline{f}(p)$ converges to 0 as $\operatorname{Re} p \to \infty$.*

Proof. Note that for $\operatorname{Re} t > 0$,

$$\begin{aligned}
\left|\overline{f}(p_0+t)\right| &= \left|\int_0^\infty f(x)\mathrm{e}^{-(p_0+t)x}\,\mathrm{d}x\right| \\
&\leqslant \left|\int_0^1 f(x)\mathrm{e}^{-p_0x}\mathrm{e}^{-tx}\,\mathrm{d}x\right| + \left|\int_1^\infty f(x)\mathrm{e}^{-p_0x}\mathrm{e}^{-tx}\,\mathrm{d}x\right| \\
&\leqslant \int_0^1 \left|f(x)\mathrm{e}^{-p_0x}\mathrm{e}^{-tx}\right|\,\mathrm{d}x + \left|\int_1^\infty f(x)\mathrm{e}^{-p_0x}\mathrm{e}^{-tx}\,\mathrm{d}x\right| \\
&= M\int_0^1 \mathrm{e}^{-(\operatorname{Re} t)x}\,\mathrm{d}x + \mathrm{e}^{-\operatorname{Re} t}\int_1^\infty \left|f(x)\mathrm{e}^{-p_0x}\right|\,\mathrm{d}x \qquad \left[\,\left|f(x)\mathrm{e}^{-p_0x}\right| \leqslant M \text{ on } [0,1]\,\right] \\
&= M\left|\left(\frac{1-\mathrm{e}^{-\operatorname{Re} t}}{\operatorname{Re} t}\right)\right| + \mathrm{e}^{-\operatorname{Re} t}\int_1^\infty \left|f(x)\mathrm{e}^{-p_0x}\right|\,\mathrm{d}x \\
&\to 0 \text{ as } \operatorname{Re} t \to \infty
\end{aligned}$$

(The requirement that f be continuous is not necessary, but is not a restrictive hypothesis and simplifies the proof substantially. A more careful calculation (in which one replaces the limit of integration '1' above with ϵ and then lets $\epsilon \to 0$) shows that the asymptotic behaviour of the Laplace transform is $f(0)/p$ as $p \to \infty$, a result known as Watson's Lemma.) ■

Note also the following property of the Laplace transform.

Proposition 43 *Let $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$ and assume $\bar{f}(p)$ converges on some half-plane $\operatorname{Re} p > c$. Then the function $g(x) = f(x)\mathrm{e}^{-ax}$ has transform*

$$\bar{g}(p) = \bar{f}(p+a).$$

Proof. Simply note that

$$\bar{g}(p) = \int_0^\infty f(x)\mathrm{e}^{-ax}\mathrm{e}^{-px}\,\mathrm{d}x = \int_0^\infty f(x)\mathrm{e}^{-(a+p)x}\,\mathrm{d}x = \bar{f}(p+a).$$

■

For the Laplace transform to be of use treating differential equations, it needs to handle derivatives well and this is indeed the case.

Proposition 44 *Provided the Laplace transforms of $f'(x)$ and $f(x)$ converge for $\operatorname{Re}(p) > c$, and provided $f(x)\mathrm{e}^{-px} \to 0$ as $x \to \infty$ with p in the region of convergence, then*

$$\overline{f'}(p) = p\overline{f}(p) - f(0).$$

Proof. We have, by integration by parts,

$$\overline{f'}(p) = \int_0^\infty f'(x)\mathrm{e}^{-px}\,\mathrm{d}x = \left[f(x)\mathrm{e}^{-px}\right]_0^\infty - \int_0^\infty f(x)\left(-p\mathrm{e}^{-px}\right)\,\mathrm{d}x = (0 - f(0)) + p\overline{f}(p).$$

■

Corollary 45 *Provided that the Laplace transforms of $f(x), f'(x), f''(x)$ converge, and with good behaviour at infinity as above, then*

$$\overline{f''}(p) = p^2\overline{f}(p) - pf(0) - f'(0).$$

Proof. If we write $g(x) = f'(x)$ then

$$\overline{f''}(p) = \overline{g'}(p) = p\bar{g}(p) - g(0) = p\overline{f'}(p) - f'(0) = p^2\overline{f}(p) - pf(0) - f'(0).$$

■

Example 46 *The function $f(x)$ is the solution of the differential equation*

$$f''(x) - 3f'(x) + 2f(x) = 0, \qquad f(0) = f'(0) = 1.$$

Determine $\overline{f}(p)$.

Proof. Transforming the differential equation, and recalling that $f(0) = f'(0) = 1$ we see

$$\left(p^2\overline{f}(p) - p - 1\right) - 3(p\overline{f}(p) - 1) + 2\overline{f}(p) = 0.$$

Hence, rearranging this equation, we find

$$\left(p^2 - 3p + 2\right)\overline{f}(p) = p - 2$$

and hence

$$\overline{f}(p) = \frac{p-2}{(p-1)(p-2)} = \frac{1}{p-1}.$$

■

We of course recognise $\overline{f}(p)$ as the Laplace Transform of e^x. Can we reasonably write that $f(x) = \mathrm{e}^x$? For now we merely claim:

- The Laplace transform $\mathcal{L}$ is injective. So if $\bar{f}(p) = \bar{g}(p)$ on some half-plane $\operatorname{Re} p > c$ we can conclude that $f(x) = g(x)$.

We shall prove a version of this fact in due course, but for now this claim allows us to invert a range of transforms by inspection.

Example 47 *Find the Laplace inverses of*

$$\overline{f}(p) = \frac{1}{p^2(p+1)}; \qquad \bar{g}(p) = \frac{1}{p^2+2p+4}.$$

Solution. Both the given functions are not instantly recognizable, given the examples we have seen so far, but with some simple algebraic rearrangement we can quickly circumvent this. Firstly using partial fractions we see that

$$\overline{f}(p) = \frac{1}{p^2(p+1)} = \frac{1}{p^2} - \frac{1}{p} + \frac{1}{p+1}$$

and inverting the Laplace Transform we see that

$$f(x) = x - 1 + \mathrm{e}^{-x}.$$

If we complete the square in the denominator of $\bar{g}(p)$ we also see

$$\bar{g}(p) = \frac{1}{p^2+2p+4} = \frac{1}{(p+1)^2+3}.$$

Now we know $\sin(\sqrt{3}x)$ transforms to $\sqrt{3}/\left(p^2+3\right)$ and that $\mathrm{e}^{-x}h(x)$ transforms to $\bar{h}(p+1)$ and so

$$g(x) = \frac{1}{\sqrt{3}}\mathrm{e}^{-x}\sin\sqrt{3}x.$$

■

Example 48 *Find the Laplace inverse of* $p^{-2}\mathrm{e}^{-p}$.

Solution. We know that $H(x-1)$ has transform $p^{-1}\mathrm{e}^{-p}$ and we also know that x transforms to p^{-2}, so perhaps we can combine these facts somehow. Note the function $f(x) = xH(x-1)$ transforms to

$$\overline{f}(p) = \int_0^\infty xH(x-1)\mathrm{e}^{-px}\,\mathrm{d}x = \int_1^\infty x\mathrm{e}^{-px}\,\mathrm{d}x = \left[-p^{-2}(px+1)\mathrm{e}^{-px}\right]_1^\infty = p^{-2}(p+1)\mathrm{e}^{-p},$$

which is close to what we want. In fact we can see that we are wrong precisely by the transform of $H(x-1)$. So we see that the inverse transform of $p^{-2}\mathrm{e}^{-p}$ is

$$xH(x-1) - H(x-1) = (x-1)H(x-1).$$

■

It might be easiest to think of $(x-1)H(x-1)$ in terms of its graph. It is just the graph of x translated one to the right, taking value 0 on the interval $0 < x \leqslant 1$. This is a particular instance of the following result.

Proposition 49 *Assuming the Laplace transform* $\overline{f}(p)$ *of* $f(x)$ *to exist on some half-plane* $\operatorname{Re} p > c$*, and* $a > 0$ *then*

$$g(x) = f(x-a)H(x-a)$$

has transform $\bar{g}(p) = \bar{f}(p)\mathrm{e}^{-ap}$.

Proof. We have

$$\begin{aligned}
\bar{g}(p) &= \int_0^\infty f(x-a)H(x-a)\mathrm{e}^{-px}\,\mathrm{d}x \\
&= \int_a^\infty f(x-a)\mathrm{e}^{-px}\,\mathrm{d}x \\
&= \int_0^\infty f(u)\mathrm{e}^{-p(u+a)}\,\mathrm{d}u \qquad [u = x-a] \\
&= \mathrm{e}^{-ap}\int_0^\infty f(u)\mathrm{e}^{-pu}\,\mathrm{d}u \\
&= \mathrm{e}^{-ap}\bar{f}(p).
\end{aligned}$$

■

Example 50 *Solve the ODE*

$$f''(x) + 4f'(x) + 8f(x) = x, \qquad f(0) = 1, \quad f'(0) = 0.$$

Solution. Applying the transform we have

$$\left\{p^2\bar{f}(p) - p\right\} + 4\left\{p\bar{f}(p) - 1\right\} + 8\bar{f}(p) = \frac{1}{p^2},$$

and rearranging gives

$$(p^2 + 4p + 8)\bar{f}(p) - p - 4 = \frac{1}{p^2}.$$

So $\bar{f}(p)$ equals

$$\frac{p^3 + 4p^2 + 1}{p^2\left(p^2 + 4p + 8\right)} = \frac{1}{16}\left\{\frac{2}{p^2} - \frac{1}{p} + \frac{17p + 66}{p^2 + 4p + 8}\right\} = \frac{1}{16}\left\{\frac{2}{p^2} - \frac{1}{p} + \frac{17(p+2) + 32}{(p+2)^2 + 4}\right\}$$

which by inspection inverts to

$$f(x) = \frac{1}{16}\left\{2x - 1 + 17\mathrm{e}^{-2x}\cos 2x + 16\mathrm{e}^{-2x}\sin 2x\right\}.$$

We made use here of

$$\cos ax \overset{\mathcal{L}}{\mapsto} \frac{p}{p^2 + a^2}, \qquad \sin ax \overset{\mathcal{L}}{\mapsto} \frac{a}{p^2 + a^2}, \qquad f(x)\mathrm{e}^{-ax} \overset{\mathcal{L}}{\mapsto} \bar{f}(p+a).$$

■

Example 51 *Let $\tau_2 > \tau_1 > 0$. Solve the ODE*

$$x''(t) + x(t) = \delta(t-\tau_1) - \delta(t-\tau_2), \qquad x(0) = 0, \quad x'(0) = 0.$$

Solution. Applying the transform we find

$$(p^2+1)\bar{x} = \mathrm{e}^{-p\tau_1} - \mathrm{e}^{-p\tau_2}, \qquad \Longrightarrow \qquad \bar{x} = \frac{\mathrm{e}^{-p\tau_1} - \mathrm{e}^{-p\tau_2}}{p^2+1},$$

which inverts to

$$x(t) = \sin(t-\tau_1)H(t-\tau_1) - \sin(t-\tau_2)H(t-\tau_2) = \begin{cases} 0 & t < \tau_1, \\ \sin(t-\tau_1) & \tau_1 \leqslant t \leqslant \tau_2, \\ \sin(t-\tau_1) - \sin(t-\tau_2) & \tau_2 < t. \end{cases}$$

■

Unfortunately transforming a differential equation can sometimes lead to another differential equation. This is because of the following property.

Proposition 52 *Assuming that the Laplace transforms of $f(x)$ and $g(x) = xf(x)$ converge then*

$$\bar{g}(p) = -\frac{\mathrm{d}\overline{f}}{\mathrm{d}p}.$$

Proof. Using *differentiation under the integral sign*

$$\frac{\mathrm{d}\overline{f}}{\mathrm{d}p} = \frac{\mathrm{d}}{\mathrm{d}p}\int_0^\infty f(x)\mathrm{e}^{-px}\,\mathrm{d}x = \int_0^\infty \frac{\partial}{\partial p}\left(f(x)\mathrm{e}^{-px}\right)\,\mathrm{d}x = -\int_0^\infty xf(x)\mathrm{e}^{-px}\,\mathrm{d}x = -\bar{g}(p).$$

■

Example 53 *Find the inverse Laplace transform of $(p-a)^{-n}$.*

Solution. From Example 36 we know that the Laplace transform of x^n is $n!p^{-n-1}$. Hence by Proposition 43 we can see that the $x^{n-1}\mathrm{e}^{ax}/(n-1)!$ has transform $(p-a)^{-n}$.

Alternaitvely we could make use of Proposition 52 and Example 35 to note

$$\begin{aligned} (p-a)^{-n} &= \frac{1}{(n-1)!}\left(-\frac{\mathrm{d}}{\mathrm{d}p}\right)^{n-1}(p-a)^{-1} = \frac{1}{(n-1)!}\left(-\frac{\mathrm{d}}{\mathrm{d}p}\right)^{n-1}\mathcal{L}(\mathrm{e}^{ax}) \\ &= \frac{1}{(n-1)!}\mathcal{L}(x^{n-1}\mathrm{e}^{ax}) = \mathcal{L}\left(\frac{x^{n-1}\mathrm{e}^{ax}}{(n-1)!}\right). \end{aligned}$$

■

Example 54 *Find the inverse Laplace transform of* $(p^2+2p+2)^{-2}$.

Solution. We know that

$$\sin x \overset{\mathcal{L}}{\mapsto} \frac{1}{p^2+1}; \qquad \cos x \overset{\mathcal{L}}{\mapsto} \frac{p}{p^2+1}.$$

So

$$x\sin x \overset{\mathcal{L}}{\mapsto} -\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1}{p^2+1}\right) = \frac{2p}{(p^2+1)^2}; \qquad x\cos x \overset{\mathcal{L}}{\mapsto} -\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{p}{p^2+1}\right) = \frac{p^2-1}{(p^2+1)^2}.$$

So we might write

$$\begin{aligned}\frac{1}{(p^2+1)^2} &= \frac{1}{2}\left\{\frac{p^2+1}{(p^2+1)^2} - \frac{p^2-1}{(p^2+1)^2}\right\} \\ &= \frac{1}{2}\left\{\frac{1}{p^2+1} - \frac{p^2-1}{(p^2+1)^2}\right\} \overset{\mathcal{L}^{-1}}{\mapsto} \frac{1}{2}\left(\sin x - x\cos x\right).\end{aligned}$$

Thus

$$\frac{1}{(p^2+2p+2)^2} = \frac{1}{\left((p+1)^2+1\right)} \overset{\mathcal{L}^{-1}}{\mapsto} \frac{\mathrm{e}^{-x}}{2}\left(\sin x - x\cos x\right).$$

■

Example 55 ***Bessel's function of order zero***, $J_0(x)$, *satisfies the initial-value problem*

$$x\frac{\mathrm{d}^2 J_0}{\mathrm{d}x^2} + \frac{\mathrm{d}J_0}{\mathrm{d}x} + xJ_0 = 0, \qquad J_0(0) = 1, \quad J_0'(0) = 0.$$

Show that $\overline{J_0}(p) = (1+p^2)^{-1/2}$.

Solution. By Propositions 44 and 52, when we apply the Laplace transform to both sides of the above IVP we get

$$-\frac{\mathrm{d}}{\mathrm{d}p}(p^2\overline{J_0} - p) + (p\overline{J_0}(p) - 1) - \frac{\mathrm{d}\overline{J_0}}{\mathrm{d}p} = 0.$$

Simplifying we see

$$(p^2+1)\frac{\mathrm{d}\overline{J_0}}{\mathrm{d}p} + p\overline{J_0} = 0.$$

This equation is separable and we may solve it to find

$$\overline{J_0}(p) = A(1+p^2)^{-1/2}$$

where A is some constant. We might try to determine A by recalling that $\overline{J_0}(p)$ approaches 0 as $\mathrm{Re}\, p$ becomes large, however this is the case for all values of A. Instead we can note that

$$\overline{J_0'}(p) = p\overline{J_0}(p) - J_0(0) = Ap(1+p^2)^{-1/2} - 1 = A(1+p^{-2})^{-1/2} - 1, \tag{2.3}$$

must also approach 0 as p becomes large. As (2.3) approaches to $A-1$ for large p then $A = 1$ and $\overline{J_0}(p) = (1+p^2)^{-1/2}$. ■

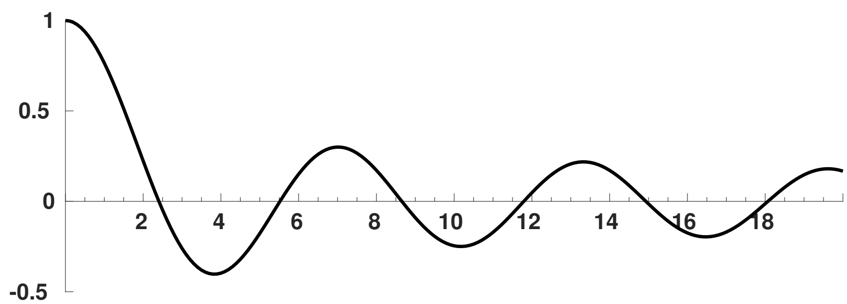


Figure 2.1: $J_0(x)$ for $0 \le x \le 20$.

We conclude with a table of transforms that we have so far determined.

$f(x)$	$\bar{f}(p)$	$f(x)$	$\bar{f}(p)$
x^n	$n!/p^{n+1}$	$f'(x)$	$p\bar{f}(p) - f(0)$
e^{ax}	$(p-a)^{-1}$	$f''(x)$	$p^2\bar{f}(p) - pf(0) - f'(0)$
$\cos ax$	$p/(p^2+a^2)$	$xf(x)$	$-\mathrm{d}\bar{f}/\mathrm{d}p$
$\sin ax$	$a/(p^2+a^2)$	$f(x-a)H(x-a)$	$\mathrm{e}^{-ap}\bar{f}(p)$
$\delta(x-a)$	e^{-ap}	$\mathrm{e}^{-ax}f(x)$	$\bar{f}(p+a)$

Chapter 3

Convolution and Inversion

Definition 56 *Given two functions* f, g *whose Laplace transforms* $\bar{f}, \bar{g}$ *exist for* $\operatorname{Re} p > c$*, we define the* ***convolution*** $h = f * g$ *by*

$$h(x) = (f * g)(x) = \int_0^x f(t)g(x-t)\,\mathrm{d}t \qquad \text{for } x \geqslant 0.$$

Remark 57 *Note that*

$$\begin{aligned}(f * g)(x) &= \int_0^x f(t)g(x-t)\,\mathrm{d}t \\ &= \int_x^0 f(x-u)g(u)\,(-\mathrm{d}u) \qquad [u = x-t] \\ &= \int_0^x g(u)f(x-u)\,\mathrm{d}u \\ &= (g * f)(x).\end{aligned}$$

Example 58 *Let* $f(x) = \sin x$ *and* $g(x) = \sin x$*. Then we have*

$$\begin{aligned}h(x) &= \int_0^x \sin t \sin(x-t)\ \mathrm{d}t \\ &= \frac{1}{2}\int_0^x \{\cos(2t-x) - \cos x\}\ \mathrm{d}t \\ &= \frac{1}{2}\left[\frac{1}{2}\sin(2t-x) - t\cos x\right]_0^x \\ &= \frac{1}{2}\left\{\frac{1}{2}\sin x + \frac{1}{2}\sin x - x\cos x\right\} \\ &= \frac{1}{2}(\sin x - x\cos x).\end{aligned}$$

We previously met this as the Laplace inverse of

$$\frac{1}{(p^2+1)^2} = \bar{f}(p)\,\bar{g}(p).$$

Example 59 *Let $f(x) = \mathrm{e}^{ax}$ and $g(x) = \mathrm{e}^{bx}$ where $a \neq b$. Then*

$$\begin{aligned} h(x) &= \int_0^x \mathrm{e}^{at}\mathrm{e}^{b(x-t)}\,\mathrm{d}t \\ &= \mathrm{e}^{bx}\int_0^x \mathrm{e}^{(a-b)t}\,\mathrm{d}t \\ &= \mathrm{e}^{bx}\left[\frac{\mathrm{e}^{(a-b)t}}{a-b}\right]_0^x \\ &= \frac{\mathrm{e}^{ax}-\mathrm{e}^{bx}}{a-b}. \end{aligned}$$

This transforms to

$$\begin{aligned} \bar{h}(p) &= \frac{1}{a-b}\left\{\frac{1}{p-a}-\frac{1}{p-b}\right\} \\ &= \frac{1}{a-b}\left\{\frac{a-b}{(p-a)(p-b)}\right\} \\ &= \frac{1}{(p-a)(p-b)} \\ &= \bar{f}\,(p)\bar{g}(p). \end{aligned}$$

Consequently the following theorem should not come as a great surprise.

Theorem 60 *Given two functions f and g whose Laplace transforms $\bar{f}$ and $\bar{g}$ exist for* $\mathrm{Re}\, p > c$. *Then*

$$\bar{h} = \bar{f}\,\bar{g}$$

where $h = f * g$.

Proof.

$$\begin{aligned}
\bar{f}(p)\bar{g}(p) &= \left(\int_0^\infty f(t)\mathrm{e}^{-pt}\,\mathrm{d}t\right)\left(\int_0^\infty g(x)\mathrm{e}^{-px}\,\mathrm{d}x\right) \\
&= \int_0^\infty \int_0^\infty f(t)g(x)\mathrm{e}^{-p(x+t)}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\infty \int_t^\infty f(t)g(y-t)\mathrm{e}^{-py}\,\mathrm{d}y\,\mathrm{d}t \qquad [x = y-t] \\
&= \iint_R f(t)g(y-t)\mathrm{e}^{-py}\,\mathrm{d}y\,\mathrm{d}t,
\end{aligned}$$

where R is the region

$$R = \{(y,t) : y \geqslant t \geqslant 0\}.$$

If we swap the order of integration we instead find

$$\begin{aligned}
\bar{f}(p)\bar{g}(p) &= \int_0^\infty \int_0^y f(t)g(y-t)\mathrm{e}^{-py}\,\mathrm{d}t\,\mathrm{d}y \\
&= \int_0^\infty \left\{\int_0^y f(t)g(y-t)\,\mathrm{d}t\right\} \mathrm{e}^{-py}\,\mathrm{d}y \\
&= \int_0^\infty (f*g)(y)\,\mathrm{e}^{-py}\,\mathrm{d}y \\
&= \bar{h}(p).
\end{aligned}$$

■

Example 61 *Find the Laplace inverse of*

$$\bar{f}(p) = \frac{p}{(p^2+1)^2}.$$

Solution. Note that $\bar{f}(p)$ is the product of

$$\overline{\cos}(p) = \frac{p}{p^2+1} \qquad \text{and} \qquad \overline{\sin}(p) = \frac{1}{p^2+1}.$$

Hence by the Convolution Theorem

$$\begin{aligned}
f(x) &= \int_0^x \sin t \cos(x-t)\,\mathrm{d}t \\
&= \left\{\cos x \int_0^x \sin t \cos t\,\mathrm{d}t + \sin x \int_0^x \sin^2 t\,\mathrm{d}t\right\} \\
&= \frac{1}{2}\left\{\cos x \int_0^x \sin 2t\,\mathrm{d}t + \sin x \int_0^x (1-\cos 2t)\ \mathrm{d}t\right\} \\
&= \frac{1}{2}\left\{\cos x \left[\frac{1}{2} - \frac{1}{2}\cos 2x\right] + \sin x \left[x - \frac{1}{2}\sin 2x\right]\right\} \\
&= \frac{1}{4}\{\cos x - \cos x \cos 2x + 2x \sin x - \sin x \sin 2x\} \\
&= \frac{1}{4}\left\{\cos x - \cos x\left(1 - 2\sin^2 x\right) + 2x\sin x - 2\sin^2 x \cos x\right\} = \frac{1}{2} x \sin x.
\end{aligned}$$

Having determined this answer we see we could also have realized this as

$$\frac{p}{(p^2+1)^2} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1}{p^2+1}\right) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}p}\left(\overline{\sin}(p)\right) = \frac{1}{2}\mathcal{L}\left(x \sin x\right).$$

■

Example 62 *Determine the solution of the IVP*

$$y''(x) + 3y'(x) + 2y(x) = f(x), \qquad y(0) = y'(0) = 1,$$

with your solution involving a convolution.

Solution. Applying the Laplace transform we find

$$\left\{p^2\bar{y} - p - 1\right\} + 3\left\{p\bar{y} - 1\right\} + 2\bar{y} = \bar{f}.$$

Hence

$$(p+2)(p+1)\bar{y} = p + 4 + \bar{f},$$

and

$$\begin{aligned}
\bar{y} &= \frac{p+4}{(p+2)(p+1)} + \frac{\bar{f}}{(p+2)(p+1)} \\
&= \frac{3}{p+1} - \frac{2}{p+2} + \left(\frac{1}{p+1} - \frac{1}{p+2}\right)\bar{f}.
\end{aligned}$$

Hence

$$\bar{y}(p) = 3\mathrm{e}^{-x} - 2\mathrm{e}^{-2x} + \int_0^x \left(\mathrm{e}^{-t} - \mathrm{e}^{-2t}\right) f(x-t)\,\mathrm{d}t.$$

■

We now have a various toolkit for finding inverse Laplace transforms using some form of inspection. However we have no comprehensive method for achieving this, nor even certainty yet that the Laplace transform is injective. We first prove this last fact for a fairly wide range of functions.

Theorem 63 ***(Injectivity of the Laplace Transform)*** *Let f be a continuous function on $[0,\infty)$, bounded by some function $M\mathrm{e}^{cx}$ and such that $\bar{f}(p) = 0$ for $\operatorname{Re} p > c$. Then $f = 0$.*

Proof. (Non-examinable) We will need to make use of the *Weierstrass Approximation Theorem* during this proof which says that given any continuous function on a closed bounded interval there is a sequence of polynomials that uniformly converge to it.

Fix $k > c$ and set $p = k + n + 1$ where $n \geqslant 0$. We then have that

$$\begin{aligned} 0 &= \int_0^\infty f(x)\mathrm{e}^{-px}\,\mathrm{d}x \\ &= \int_0^\infty \mathrm{e}^{-nx}\mathrm{e}^{-kx}\mathrm{e}^{-x} f(x)\,\mathrm{d}x \\ &= \int_0^1 y^n y^k f(-\log y)\,\mathrm{d}y \qquad [y = \mathrm{e}^{-x}] \\ &= \int_0^1 y^n g(y)\,\mathrm{d}y \end{aligned}$$

where $g(y) = y^k f(-\log y)$. Note that g is immediately continuous on $(0,1]$ and is also continuous at 0 as

$$|g(y)| = \left|y^k f(-\log y)\right| = \mathrm{e}^{-kx}\,|f(x)| \leqslant M\mathrm{e}^{(c-k)x} \to 0 \quad \text{as } y \to 0 \text{ (or } x \to \infty).$$

By linearity it follows that

$$\int_0^1 p(y)g(y)\,\mathrm{d}y = 0$$

for any polynomial p. If p_n is a sequence of polynomials uniformly converging to g then we have

$$\int_0^1 g(y)^2\,\mathrm{d}y = \lim_{n\to\infty} \int_0^1 p_n(y)g(y)\,\mathrm{d}y = 0$$

and hence $g = 0$ and finally $f = 0$. ■

And at last we arrive at the Inversion Theorem.

Theorem 64 ***(Inversion Theorem for Laplace Transform)*** *Let f be a differentiable function on $(0,\infty)$ such that $\bar{f}(p)$ exists for $\operatorname{Re} p > c$. Then for $x > 0$,*

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \bar{f}(p) \mathrm{e}^{px}\, \mathrm{d}p \qquad (\sigma > c)\,.$$

Remark 65 *A complete proof of this result would be technical and beyond the scope of this course. However we shall prove the above result for rational functions which in practice addresses the inverse problem for many of the functions that we have already met. We shall revisit this inversion theorem when we tackle the inverse problem for the Fourier transform.*

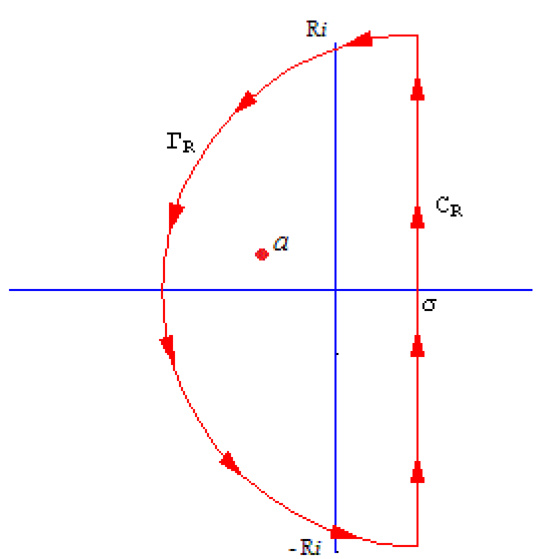
Proof. (Non-examinable.) Suppose that $f(x)$ has a Laplace transform $\bar{f}(p)$ which is a rational function of the form

$$\bar{f}(p) = \frac{g(p)}{(p-a)^n}$$

where g is a polynomial of degree less than n. Consider the integral

$$I(x) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \bar{f}(p) \mathrm{e}^{px}\, \mathrm{d}p$$

where $x > 0$ and $\sigma > \operatorname{Re} a$. We will seek to evaluate this integral using the Residue Theorem applied to the contour shown in the figure:



This gives

$$\frac{1}{2\pi \mathrm{i}}\int_{C_R}\bar{f}(p)\mathrm{e}^{px}\,\mathrm{d}p+\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_R}\bar{f}(p)\mathrm{e}^{px}\,\mathrm{d}p=\operatorname{res}(\bar{f}(p)\mathrm{e}^{px};a).$$

We also know

$$\operatorname{res}(\bar{f}(p)\mathrm{e}^{px};a)=\operatorname{res}\left(\frac{g(p)\mathrm{e}^{px}}{(p-a)^n};a\right)=\frac{1}{(n-1)!}\left.\frac{\mathrm{d}^{n-1}}{\mathrm{d}p^{n-1}}\right|_{p=a}g(p)\mathrm{e}^{px}.$$

As the degree of g is less than n then $\left|\bar{f}(p)\right|=O(|p|^{-1})$ for suitably large $|p|$ and so for suitably large R we have that

$$\begin{aligned}
\left|\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_R}\bar{f}(p)\mathrm{e}^{px}\,\mathrm{d}p\right| &\leqslant \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}\left|\bar{f}(\sigma+R\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{x(\sigma+R\mathrm{e}^{\mathrm{i}\theta})}\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\right|\mathrm{d}\theta \qquad [p=\sigma+R\mathrm{e}^{\mathrm{i}\theta}]\\
&\leqslant \frac{R\mathrm{e}^{x\sigma}}{2\pi}O\left(\frac{1}{R}\right)\int_{\pi/2}^{3\pi/2}\mathrm{e}^{xR\cos\theta}\,\mathrm{d}\theta\\
&= O(1)\int_{\pi/2}^{\pi}\mathrm{e}^{xR\cos\theta}\,\mathrm{d}\theta\\
&= O(1)\int_{0}^{\pi/2}\mathrm{e}^{-xR\sin\theta}\,\mathrm{d}\theta\\
&= O(1)\int_{0}^{\pi/2}\mathrm{e}^{-2xR\theta/\pi}\,\mathrm{d}\theta \qquad [\text{by Jordan's Lemma}]\\
&= O\left(R^{-1}\right)\to 0 \qquad \text{as } R\to\infty.
\end{aligned}$$

Hence letting $R\to\infty$ we have

$$I(x)=\frac{1}{2\pi \mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}\bar{f}(p)\mathrm{e}^{px}\,\mathrm{d}p=\frac{1}{(n-1)!}\left.\frac{\mathrm{d}^{n-1}}{\mathrm{d}p^{n-1}}\right|_{p=a}g(p)\mathrm{e}^{px}.$$

This should be an expression for $f(x)$. Given the injectivity of the Laplace transform, it is sufficient to show that this does indeed transform into $\bar{f}(p)$. We shall demonstrate this using Leibniz's rule for differentiating products. We have

$$\begin{aligned}
\bar{I}(p) &= \frac{1}{(n-1)!}\int_0^\infty \left(\left.\frac{\mathrm{d}^{n-1}}{\mathrm{d}p^{n-1}}\right|_{p=a} g(p)\mathrm{e}^{px} \right) \mathrm{e}^{-px}\,\mathrm{d}x \\
&= \frac{1}{(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{k}\int_0^\infty g^{(k)}(a)x^{n-1-k}\mathrm{e}^{ax}\mathrm{e}^{-px}\,\mathrm{d}x \\
&= \frac{1}{(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{k}g^{(k)}(a)\int_0^\infty x^{n-1-k}\mathrm{e}^{-(p-a)x}\,\mathrm{d}x \\
&= \frac{1}{(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{k}g^{(k)}(a)\frac{(n-1-k)!}{(p-a)^{n-k}} \\
&= \frac{1}{(p-a)^n}\sum_{k=0}^{n-1}\frac{g^{(k)}(a)}{k!}(p-a)^k \\
&= \frac{g(p)}{(p-a)^n} = \bar{f}(p),
\end{aligned}$$

noting the last sum is just Taylor expansion of the polynomial of $g(p)$ centred at a and so equals $g(p)$. By the injectivity of $\mathcal{L}$ we can conclude that $I(x) = f(x)$.

As a rational function (with a numerator of degree strictly less than its denominator) can be written as a linear combination of such $\bar{f}$ then the desired result follows by linearity. ■

Example 66 *Find the Laplace inverse of*

$$\bar{f}(p) = \frac{1}{(p^2+1)^2}.$$

Solution. Consider the contour described in the remark following the Inversion Theorem. For $x > 0$ we will calculate

$$I = \frac{1}{2\pi\mathrm{i}}\int_\gamma \bar{f}(p)\mathrm{e}^{px}\,\mathrm{d}p = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\mathrm{e}^{px}}{(p^2+1)^2}\,\mathrm{d}p$$

around the contour $\gamma = C_R \cup \Gamma_R$. The integrand has double poles at $\pm\mathrm{i}$ and so by Cauchy's Residue Theorem I equals

$$I = \mathrm{res}\left(\frac{\mathrm{e}^{px}}{(p^2+1)^2};\mathrm{i}\right) + \mathrm{res}\left(\frac{\mathrm{e}^{px}}{(p^2+1)^2};-\mathrm{i}\right).$$

Now

$$\begin{aligned}\operatorname{res}\left(\frac{\mathrm{e}^{px}}{(p^2+1)^2};\mathrm{i}\right) &= \operatorname{res}\left(\frac{\mathrm{e}^{px}}{(p+\mathrm{i})^2\,(p-\mathrm{i})^2};\mathrm{i}\right)\\ &= \frac{1}{1!}\left.\frac{\mathrm{d}}{\mathrm{d}p}\right|_{p=\mathrm{i}}\frac{\mathrm{e}^{px}}{(p+\mathrm{i})^2}\\ &= \left.\left\{\frac{x\mathrm{e}^{px}}{(p+\mathrm{i})^2}-\frac{2\mathrm{e}^{px}}{(p+\mathrm{i})^3}\right\}\right|_{p=\mathrm{i}}\\ &= \frac{-x\mathrm{e}^{\mathrm{i}x}}{4}-\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}x}}{4}.\end{aligned}$$

Similarly

$$\operatorname{res}\left(\frac{\mathrm{e}^{px}}{(p^2+1)^2};-\mathrm{i}\right)=\frac{1}{1!}\left.\frac{\mathrm{d}}{\mathrm{d}p}\right|_{p=-\mathrm{i}}\frac{\mathrm{e}^{px}}{(p-\mathrm{i})^2}=\frac{-x\mathrm{e}^{-\mathrm{i}x}}{4}+\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}x}}{4}.$$

Hence

$$\begin{aligned}I &= \frac{1}{4}\left(\mathrm{i}(\mathrm{e}^{-\mathrm{i}x}-\mathrm{e}^{\mathrm{i}x})-x(\mathrm{e}^{\mathrm{i}x}+\mathrm{e}^{-\mathrm{i}x})\right)\\ &= \frac{1}{4}\left(\mathrm{i}(-2\mathrm{i}\sin x)-x(2\cos x)\right)\\ &= \frac{1}{2}\left(\sin x-x\cos x\right).\end{aligned}$$

We can parametrize the Γ_R arc in γ by $z=\sigma+R\mathrm{e}^{\mathrm{i}\theta}$ where $\pi/2\leqslant\theta\leqslant 3\pi/2$. Parametrizing the semicircular arc as $p=\sigma+R\mathrm{e}^{\mathrm{i}\theta}$ and using the Estimation Theorem we have

$$\begin{aligned}\left|\int_\Gamma\frac{\mathrm{e}^{px}}{(p^2+1)^2}\,\mathrm{d}p\right| &\leqslant \frac{\pi R}{O(R^4)}\times\sup_{\theta\in(\pi/2,3\pi/2)}\left|\mathrm{e}^{x(\sigma+R\mathrm{e}^{\mathrm{i}\theta})}\right|\\ &\leqslant \frac{\pi R\mathrm{e}^{x\sigma}}{O(R^4)}\times\sup_{\theta\in(\pi/2,3\pi/2)}\mathrm{e}^{xR\cos\theta}\\ &\leqslant \frac{\pi R\mathrm{e}^{x\sigma}}{O(R^4)}\qquad[\text{as } xR\cos\theta<0]\\ &= O\left(R^{-3}\right)\to 0\qquad\text{as } R\to\infty.\end{aligned}$$

Thus

$$f(x)=\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}\bar{f}(p)\mathrm{e}^{px}\,\mathrm{d}p=\lim_{R\to\infty}\frac{1}{2\pi\mathrm{i}}\int_\gamma\bar{f}(p)\mathrm{e}^{px}\,\mathrm{d}p=\frac{1}{2}\left(\sin x-x\cos x\right).$$

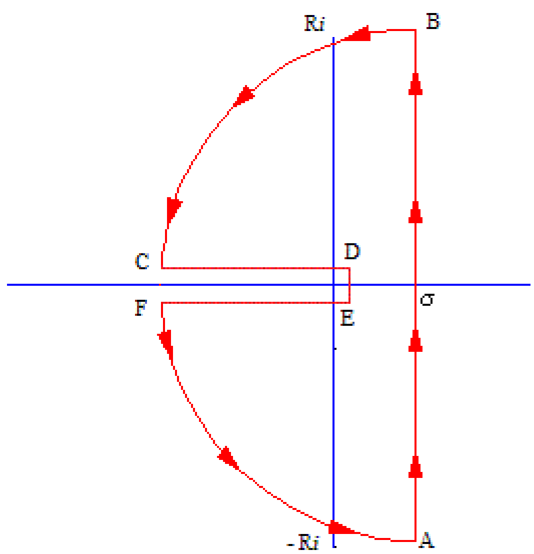
■

Example 67 *Find the Laplace inverse of* $\bar{f}(p) = p^{-1/3}$.

Solution. We define a branch of $p^{-1/3}$ in the cut plane $\mathbb{C}\backslash(-\infty, 0]$ by

$$p^{-1/3} = r^{-1/3}\mathrm{e}^{-i\theta/3} \qquad \text{where} \qquad p = r\mathrm{e}^{i\theta} \quad -\pi < \theta < \pi,$$

and adapt the contour γ around the cut, and divide it into various line segments and arcs AB, BC, CD, EF, FA as in the diagram below.



By Cauchy's Theorem we have

$$I = \frac{1}{2\pi\mathrm{i}}\int_\gamma \bar{f}(p)\mathrm{e}^{px}\,\mathrm{d}p = 0.$$

Along the topside of the cut CD we have $\theta = \pi$ and $p = -r$ and

$$p^{-1/3} = (r\mathrm{e}^{\mathrm{i}\pi})^{-1/3} = r^{-1/3}\mathrm{e}^{-\mathrm{i}\pi/3}$$

and along the bottom side of the cut EF we have $\theta = -\pi$ and $p = -r$ and

$$p^{-1/3} = (r\mathrm{e}^{-\mathrm{i}\pi})^{-1/3} = r^{-1/3}\mathrm{e}^{\mathrm{i}\pi/3}.$$

Hence when $R \to \infty$ we have

$$\frac{1}{2\pi\mathrm{i}}\int_C^D \bar{f}(p)\mathrm{e}^{px}\,\mathrm{d}p \;\to\; \frac{\mathrm{e}^{-\mathrm{i}\pi/3}}{2\pi\mathrm{i}}\int_0^\infty \frac{\mathrm{e}^{-rx}}{r^{1/3}}\,\mathrm{d}r = \frac{\mathrm{e}^{-\mathrm{i}\pi/3}}{2\pi\mathrm{i}}x^{-2/3}\int_0^\infty \frac{\mathrm{e}^{-u}}{u^{1/3}}\,\mathrm{d}u = \Gamma(2/3)\frac{\mathrm{e}^{-\mathrm{i}\pi/3}}{2\pi\mathrm{i}}x^{-2/3};$$

$$\frac{1}{2\pi\mathrm{i}}\int_E^F \bar{f}(p)\mathrm{e}^{px}\,\mathrm{d}p \;\to\; -\frac{\mathrm{e}^{\mathrm{i}\pi/3}}{2\pi\mathrm{i}}\int_0^\infty \frac{\mathrm{e}^{-rx}}{r^{1/3}}\,\mathrm{d}r = -\frac{\mathrm{e}^{\mathrm{i}\pi/3}}{2\pi\mathrm{i}}x^{-2/3}\int_0^\infty \frac{\mathrm{e}^{-u}}{u^{1/3}}\,\mathrm{d}u = -\Gamma(2/3)\frac{\mathrm{e}^{i\pi/3}}{2\pi\mathrm{i}}x^{-2/3}.$$

Together these add to

$$\frac{\Gamma(2/3)}{2\pi\mathrm{i}}x^{-2/3}\left(\mathrm{e}^{-\mathrm{i}\pi/3}-\mathrm{e}^{\mathrm{i}\pi/3}\right)=-\frac{\Gamma(2/3)\sqrt{3}}{2\pi x^{2/3}}.$$

Now considering the BC and FA integrals we have

$$\begin{aligned}\left|\int_B^C \frac{\mathrm{e}^{px}}{p^{1/3}}\,\mathrm{d}p\right| &= \frac{O(R)}{O(R^{1/3})}\int_{\pi/2}^{\pi}\mathrm{e}^{x(\sigma+R\cos\theta)}\,\mathrm{d}\theta \\ &= O\left(R^{2/3}\right)\int_0^{\pi/2}\mathrm{e}^{-xR\sin\theta}\,\mathrm{d}\theta \\ &\leqslant O\left(R^{2/3}\right)\int_0^{\pi/2}\mathrm{e}^{-2xR\theta/\pi}\,\mathrm{d}\theta \qquad \text{[by Jordan's Lemma]} \\ &= O(R^{-1/3})\to 0 \qquad \text{as } R\to\infty.\end{aligned}$$

Similarly the contribution from FA tends to zero in the limit. Hence letting $R\to\infty$ we have

$$f(x)=\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}\bar{f}(p)\mathrm{e}^{px}\,\mathrm{d}p=\frac{\Gamma(2/3)\sqrt{3}}{2\pi x^{2/3}}.$$

We already know that

$$x^a \overset{\mathcal{L}}{\longmapsto} \frac{\Gamma(a+1)}{p^{a+1}}, \qquad \text{so that} \qquad x^{-2/3}\overset{\mathcal{L}}{\longmapsto}\frac{\Gamma(1/3)}{p^{1/3}}$$

so our answer may seem somewhat wrong however it is the case[1] that

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)=\frac{2\pi}{\sqrt{3}}$$

[1] Since you ask: we prove that $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$ for z not an integer (the Gamma function has poles at the negative integers). First take $0<\operatorname{Re} z<1$. Then

$$\begin{aligned}\Gamma(z)\Gamma(1-z) &= \int_0^\infty\int_0^\infty \mathrm{e}^{-(s+t)}s^{z-1}t^z\,\mathrm{d}s\mathrm{d}t \\ &= \int_0^\infty\int_0^\infty \mathrm{e}^{-u}v^{z-1}\frac{\mathrm{d}u\mathrm{d}v}{1+v} \qquad (\text{by } u=s+t,\ v=s/t) \\ &= \int_0^\infty \frac{v^{z-1}}{1+v}\mathrm{d}v \\ &= \frac{\pi}{\sin\pi z}\end{aligned}$$

where the last integral ia a standard one round a keyhole contour with the branch cut for v^{z-1} taken along the positive real axis and the value of the integral coming from the residue of the pole at $v=-1$. The result holds for other values of z by holomorphic continuation. Put $z=1/3$ to get the result above.

and so

$$f(x) = \frac{\Gamma(2/3)\sqrt{3}}{2\pi x^{2/3}} = \frac{2\pi}{\Gamma(1/3)\sqrt{3}}\frac{\sqrt{3}}{2\pi x^{2/3}} = \frac{1}{\Gamma(1/3)x^{2/3}}.$$

■

Theorem 68 ***(Term-by-term Laplace Inversion)*** *Let f be a differentiable function on $(0,\infty)$ such that $\bar{f}(p)$ exists and is expressible as*

$$\bar{f}(p) = \sum_{n=0}^{\infty} \frac{a_n}{p^{n+1}} \qquad \text{for} \quad \operatorname{Re} p > c \geqslant 0. \tag{3.1}$$

Then

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}.$$

Proof. The series in (3.1) converges for $|p| > c$ and defines a holomorphic function in that domain. Further, as in the proof of Laurent's Theorem, the series converges uniformly on any $\gamma(0,r)$ where $r > c$ and we may also note that $\left|\bar{f}(p)\right| = O(1/|p|)$ as $p \to \infty$.

By the Inversion Theorem we have

$$\begin{aligned}
f(x) &= \frac{1}{2\pi \mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \bar{f}(p)\mathrm{e}^{px}\,\mathrm{d}p \\
&= \frac{1}{2\pi \mathrm{i}} \lim_{R\to\infty} \int_{\sigma-\mathrm{i}R}^{\sigma+\mathrm{i}R} \bar{f}(p)\mathrm{e}^{px}\,\mathrm{d}p \qquad \text{[by definition]} \\
&= \frac{1}{2\pi \mathrm{i}} \lim_{R\to\infty} \int_{\sigma-\mathrm{i}R}^{\sigma+\mathrm{i}R} \left(\sum_{n=0}^{\infty} \frac{a_n}{p^{n+1}}\mathrm{e}^{px}\right) \mathrm{d}p \\
&= \frac{1}{2\pi \mathrm{i}} \lim_{R\to\infty} \int_{\gamma} \left(\sum_{n=0}^{\infty} \frac{a_n}{p^{n+1}}\mathrm{e}^{px}\right) \mathrm{d}p \qquad \text{[using } \left|\bar{f}(p)\right| = O(1/|p|) \text{ and Jordan]} \\
&= \frac{1}{2\pi \mathrm{i}} \lim_{R\to\infty} \int_{\gamma(0,R)} \left(\sum_{n=0}^{\infty} \frac{a_n}{p^{n+1}}\mathrm{e}^{px}\right) \mathrm{d}p \qquad \text{[by Deformation Theorem]} \\
&= \frac{1}{2\pi i} \lim_{R\to\infty} \sum_{n=0}^{\infty} \int_{\gamma(0,R)} \left(\frac{a_n}{p^{n+1}}\mathrm{e}^{px}\right) \mathrm{d}p \qquad \text{[by uniform convergence on } \gamma(0,R)\text{]} \\
&= \frac{1}{2\pi \mathrm{i}} \lim_{R\to\infty} \sum_{n=0}^{\infty} \frac{2\pi \mathrm{i} a_n x^n}{n!} \qquad \text{[by Cauchy's Residue Theorem]} \\
&= \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}.
\end{aligned}$$

■

Example 69 *Find the Laplace inverse of* $(p^2+1)^{-1}$ *using term-by-term inversion.*

Solution. We have

$$\begin{aligned}\left(1+p^2\right)^{-1} &= \frac{1}{p^2}\left(1+\frac{1}{p^2}\right)^{-1} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{p^{2k+2}} \\ &\overset{\mathcal{L}^{-1}}{\longmapsto} \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!} \\ &= \sin x.\end{aligned}$$

■

Example 70 *From Example 55 we have that*

$$\overline{J_0}(p) = (1+p^2)^{-1/2}.$$

Find the Taylor series for $J_0(x)$.

Solution. We have

$$\begin{aligned}\overline{J_0}(p) &= p^{-1}(1+p^{-2})^{-1/2} \\ &= p^{-1}\sum_{k=0}^{\infty}\frac{\frac{-1}{2}\times\frac{-3}{2}\times\cdots\times\frac{1-2k}{2}}{k!}\left(\frac{1}{p^2}\right)^k \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k 1\times 3\times\cdots\times(2k-1)}{2^k k!}\frac{1}{p^{2k+1}} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{2^{2k}(k!)^2}\frac{1}{p^{2k+1}} \\ &= \sum_{k=0}^{\infty}\binom{2k}{k}\frac{(-1)^k}{2^{2k}p^{2k+1}}.\end{aligned}$$

Hence, inverting term-by-term, we have

$$J_0(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}.$$

■

Chapter 4

Fourier Transform and Applications

Definition 71 *Let $f : \mathbb{R} \to \mathbb{C}$ be integrable. Then the **Fourier transform** $\hat{f}(s)$ of $f(x)$ is*

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x.$$

We will also denote this $(\mathcal{F}f)(s)$ and write $\mathcal{F}$ for the Fourier transform.

Remark 72 *Note that a much smaller range of functions have a convergent Fourier transform compared with the Laplace transform. The multiplicand of e^{-px} in the Laplace transform means that many common functions – though not something like e^{x^2} (which is too big for large x) nor x^{-1} (which is not integrable near 0) – have a convergent Laplace transform. The requirement that f be integrable on the whole of $\mathbb{R}$ is therefore relatively restrictive.*

Remark 73 *The Fourier transform of the density function $f_X(x)$ of a random variable X is (up to a minus sign) its characteristic function: $\hat{f}_X(s) = \mathbb{E}[\mathrm{e}^{-\mathrm{i}sX}] = \phi_X(-s)$.*

Example 74 *Let $f = \mathbb{1}_{[-1,1]}$. Determine $\hat{f}$.*

Solution. We have

$$\hat{f}(s) = \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x = \frac{-1}{\mathrm{i}s}\left[\mathrm{e}^{-\mathrm{i}sx}\right]_{x=-1}^{x=1} = \frac{\mathrm{e}^{\mathrm{i}s} - \mathrm{e}^{-\mathrm{i}s}}{\mathrm{i}s} = \frac{2\sin s}{s}.$$

■

Example 75 *Let $g(x) = \mathrm{e}^{-a|x|}$ where $a > 0$. Determine $\hat{g}$.*

Solution. We have

$$\begin{aligned}
\hat{g}(s) &= \int_{-\infty}^{\infty} \mathrm{e}^{-a|x|}\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x \\
&= \int_{-\infty}^{0} \mathrm{e}^{(a-\mathrm{i}s)x}\,\mathrm{d}x + \int_{0}^{\infty} \mathrm{e}^{-(a+\mathrm{i}s)}\,\mathrm{d}x \\
&= \frac{1}{a-\mathrm{i}s}\left[\mathrm{e}^{(a-\mathrm{i}s)x}\right]_{-\infty}^{0} - \frac{1}{a+\mathrm{i}s}\left[\mathrm{e}^{-(a+\mathrm{i}s)x}\right]_{0}^{\infty} \\
&= \frac{1}{a-\mathrm{i}s} + \frac{1}{a+\mathrm{i}s} \\
&= \frac{2a}{a^2+s^2}.
\end{aligned}$$

■

Example 76 *Using the sifting property, we see that the Fourier transform of* $\delta(x-a)$ *is* e^{-isa}.

Example 77 *Let* $a > 0$*. Show that the Fourier transform of* $f(x) = \mathrm{e}^{-a^2x^2}$ *equals*

$$\hat{f}(s) = \frac{\sqrt{\pi}}{a}\exp\left(\frac{-s^2}{4a^2}\right).$$

Solution. We are interested in

$$\begin{aligned}
\hat{f}(s) &= \int_{-\infty}^{\infty} \exp(-a^2x^2)\exp(-\mathrm{i}sx)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \exp(-a^2x^2 - \mathrm{i}sx)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \exp\left(-a^2\left(x+\frac{\mathrm{i}s}{2a^2}\right)^2 - \frac{s^2}{4a^2}\right)\mathrm{d}x \\
&= \exp\left(\frac{-s^2}{4a^2}\right)\int_{-\infty}^{\infty} \exp\left(-a^2\left(x+\frac{\mathrm{i}s}{2a^2}\right)^2\right)\mathrm{d}x.
\end{aligned}$$

Let $s > 0$. We will consider the rectangular contour Γ_R with vertices $\pm R$ and $\pm R + \frac{\mathrm{i}s}{2a^2}$ and an integrand of $\exp(-a^2z^2)$. By Cauchy's Theorem

$$\int_{\Gamma_R} \exp\left(-a^2z^2\right)\,\mathrm{d}z = 0.$$

Note that the contribution from the rectangle's right and left edges satisfy

$$\left| \int_0^{s/2a^2} \exp(-a^2(\pm R + \mathrm{i}y)^2)\,\mathrm{i}\mathrm{d}y \right| \leqslant \mathrm{e}^{-a^2R^2} \int_0^{s/2a^2} \left|\exp(a^2y^2)\right| \,\mathrm{d}y \to 0 \quad \text{as } R \to \infty.$$

Hence, letting $R \to \infty$ we have

$$\int_{-\infty}^{\infty} \exp(-a^2x^2)\,\mathrm{d}x - \int_{-\infty}^{\infty} \exp\left(-a^2\left(x + \frac{\mathrm{i}s}{2a^2}\right)^2\right)\,\mathrm{d}x = 0.$$

We know (from knowledge of the normal distribution) that

$$\int_{-\infty}^{\infty} \exp(-a^2x^2)\,\mathrm{d}x = \frac{\sqrt{\pi}}{a}.$$

Hence

$$\frac{\sqrt{\pi}}{a} = \int_{-\infty}^{\infty} \exp\left(-a^2\left(x + \frac{\mathrm{i}s}{2a^2}\right)^2\right)\,\mathrm{d}x = \exp\left(\frac{s^2}{4a^2}\right)\hat{f}(s),$$

and

$$\hat{f}(s) = \frac{\sqrt{\pi}}{a}\exp\left(\frac{-s^2}{4a^2}\right).$$

■

We notice something about these examples:

$$\mathrm{e}^{-a|x|} \longleftrightarrow \frac{2a}{a^2+s^2} = 2\pi \times \frac{1}{a}\frac{1}{\pi(1+(s/a)^2)},$$
$$\mathrm{e}^{-a^2x^2} \longleftrightarrow \frac{\sqrt{\pi}}{a}\mathrm{e}^{-s^2/4a^2} = 2\pi \times \frac{1}{a}\frac{1}{\sqrt{4\pi}}\mathrm{e}^{-(s/a)^2/4}.$$

Both the right-hand sides above are continuous functions of the form $2\pi \times (1/a)g(s/a)$ with $\int_{-\infty}^{\infty} g(s)\,\mathrm{d}s = 1$. In the last question on problem sheet 1 we showed that $(1/a)g(s/a)$ tends to $\delta(s)$ as $a \to 0$. On the left, the limit is 1 in both cases. This strongly suggests the truth of the following proposition:

Proposition 78 *(Fourier transform of $\delta(x)$ and 1). We have*

$$\widehat{\delta(x)} = 1 \qquad \text{and} \qquad \widehat{1} = 2\pi\delta(s).$$

The proof of the first of these is just the action of δ on a continuous function. We are not going to prove the second (apart from the suggestion above), but we assure the reader that it is true *as a statement about the Fourier transform of a distribution*. We have not defined this concept, but it can quite straightforwardly be done after some preliminary work which we can't fit into this course. Note that we are *not* stating that e^{-isx} can be integrated in the usual sense!

We move on to consider the far-field behaviour of the Fourier transform, the transform of a derivative, and of a convolution:

Proposition 79 ***(Riemann-Lebesgue Lemma)*** *If f is an integrable function then*

$$\int_{-\infty}^{\infty} f(x)\cos sx\,\mathrm{d}x \to 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x)\sin sx\,\mathrm{d}x \to 0 \qquad \text{as } s \to \infty.$$

As a consequence

$$\hat{f}(s) \to 0 \qquad \text{as} \qquad s \to \pm\infty.$$

Proof. We shall not prove this result here. It appears in this term's *Integration* option. ■

Theorem 80 ***(Fourier Transform for Derivatives)*** *Let $f : \mathbb{R} \to \mathbb{C}$ be a differentiable function with an integrable derivative f', and*[1] *let $f(x) \to 0$ as $x \to \pm\infty$. Then*

$$(\mathcal{F}f')(s) = \mathrm{i}s(\mathcal{F}f)(s).$$

Proof. By Integration by Parts we have

$$\begin{aligned}(\mathcal{F}f')(s) &= \int_{-\infty}^{\infty} f'(x)e^{-\mathrm{i}sx}\,\mathrm{d}x \\ &= \left[f(x)e^{-\mathrm{i}sx}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-\mathrm{i}s)e^{-\mathrm{i}sx}\,\mathrm{d}x \\ &= 0 + \mathrm{i}s(\mathcal{F}f)(s)\end{aligned}$$

as required. Note that the result can be extended to derivatives of all orders (making appropriate technical assumptions on f). ■

In a similar fashion to the Laplace transform, the Fourier transform also has a convolution. However, **do note the difference in the limits**.

[1] We need f to vanish at infinity for the integration by parts. It is not enough to say that f is integrable. For example, consider a function constructed out of little bumps (each like the test function example in Chapter 1), of height 1 and area $1/n^2$, centered at $x = \pm n$ for n = 1,2,3,$\ldots$ The integral of this function is $2\sum_{n=1}^{\infty} 1/n^2$, which is finite. But the function does not tend to zero at infinity. A similar, but more complicated, condition applies to the Laplace transform of a derivative.

Theorem 81 *(**Fourier Transform Convolution**) Let f and g be integrable functions. Then the convolution $f * g$ is defined by*

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x-t)\, \mathrm{d}t$$

and is itself integrable and satisfies

$$\hat{h}(s) = \hat{f}(s)\hat{g}(s).$$

Remark 82 *As before with the Laplace convolution we now have*

$$(f * g)(x) = (g * f)(x),$$

with the proof following in a like manner.

Proof. We have

$$\begin{aligned}
\hat{f}(s)\hat{g}(s) &= \left(\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}sx}\, \mathrm{d}x\right)\left(\int_{-\infty}^{\infty} g(y)\mathrm{e}^{-\mathrm{i}sy}\, \mathrm{d}y\right) \\
&= \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} f(x)g(y)\mathrm{e}^{-\mathrm{i}s(x+y)}\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_{y=-\infty}^{\infty}\int_{u=-\infty}^{\infty} f(u-y)g(y)\mathrm{e}^{-\mathrm{i}su}\, \mathrm{d}u\, \mathrm{d}y \qquad [u = x+y] \\
&= \int_{u=-\infty}^{\infty}\left(\int_{y=-\infty}^{\infty} g(y)f(u-y)\, \mathrm{d}y\right)\mathrm{e}^{-\mathrm{i}su}\, \mathrm{d}u \\
&= \int_{u=-\infty}^{\infty} (g * f)(u)\, \mathrm{e}^{-\mathrm{i}su}\, \mathrm{d}u \\
&= \int_{u=-\infty}^{\infty} (f * g)(u)\, \mathrm{e}^{-\mathrm{i}su}\, \mathrm{d}u \\
&= \hat{h}(s).
\end{aligned}$$

■

Example 83 *Determine the convolution of:*
(i) $\delta_a(x) = \delta(x-a)$ *and* $f(x)$;
(ii) $\mathrm{e}^{-|x|}$ *with itself.*

Solution. (i) By the sifting property

$$(\delta_a * f)(x) = \int_{-\infty}^{\infty} \delta(t-a) f(x-t)\,\mathrm{d}t = f(x-a).$$

(ii) Let $f(x) = \mathrm{e}^{-|x|}$. Then

$$(f*f)(x) = \int_{-\infty}^{\infty} \mathrm{e}^{-|t|}\mathrm{e}^{-|x-t|}\,\mathrm{d}t = \int_{-\infty}^{0} \mathrm{e}^{t}\mathrm{e}^{-|x-t|}\,\mathrm{d}t + \int_{0}^{\infty} \mathrm{e}^{-t}\mathrm{e}^{-|x-t|}\,\mathrm{d}t.$$

If $x \geqslant 0$ we have

$$\begin{aligned}
(f*f)(x) &= \int_{-\infty}^{0} \mathrm{e}^{2t-x}\,\mathrm{d}t + \int_{0}^{x} \mathrm{e}^{-x}\,\mathrm{d}t + \int_{x}^{\infty} \mathrm{e}^{x-2t}\,\mathrm{d}t \\
&= \mathrm{e}^{-x}\left[\frac{\mathrm{e}^{2t}}{2}\right]_{-\infty}^{0} + x\mathrm{e}^{-x} + \mathrm{e}^{x}\left[\frac{\mathrm{e}^{-2t}}{-2}\right]_{x}^{\infty} \\
&= \frac{\mathrm{e}^{-x}}{2} + x\mathrm{e}^{-x} + \frac{\mathrm{e}^{-x}}{2} \\
&= (x+1)\mathrm{e}^{-x}.
\end{aligned}$$

If $x < 0$ we have

$$\begin{aligned}
(f*f)(x) &= \int_{-\infty}^{x} \mathrm{e}^{2t-x}\,\mathrm{d}t + \int_{x}^{0} \mathrm{e}^{x}\,\mathrm{d}t + \int_{0}^{\infty} \mathrm{e}^{x-2t}\,\mathrm{d}t \\
&= \mathrm{e}^{-x}\left[\frac{\mathrm{e}^{2t}}{2}\right]_{-\infty}^{x} - x\mathrm{e}^{x} + \mathrm{e}^{x}\left[\frac{\mathrm{e}^{-2t}}{-2}\right]_{0}^{\infty} \\
&= \frac{\mathrm{e}^{x}}{2} - x\mathrm{e}^{x} + \frac{\mathrm{e}^{x}}{2} \\
&= (1-x)\mathrm{e}^{x}.
\end{aligned}$$

Hence, putting these functions together, we see

$$(f*f)(x) = (1+|x|)\mathrm{e}^{-|x|}.$$

■

Theorem 84 ***(Inversion Theorem for Fourier Transform)*** *Let f be integrable and differentiable. Then*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}s.$$

Proof. (Sketch proof) One can demonstrate the inversion formula for the Fourier transform in various ways, but at the heart of each such proof will essentially be the identity

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x = 2\pi\delta(s) \tag{4.1}$$

which we stated as Proposition 78. **If** the inversion theorem is true after all, we would expect this as the Fourier transform of $\delta(x)$ is 1. Assuming (4.1) for now, we can argue as follows:

$$\begin{aligned}
\frac{1}{2\pi}\int_{s=-\infty}^{\infty} \hat{f}(s)\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}s &= \frac{1}{2\pi}\int_{s=-\infty}^{\infty}\left(\int_{y=-\infty}^{\infty} f(y)\mathrm{e}^{-\mathrm{i}ys}\,\mathrm{d}y\right)\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}s \\
&= \frac{1}{2\pi}\int_{y=-\infty}^{\infty} f(y)\left(\int_{s=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}s(y-x)}\,\mathrm{d}s\right)\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{y=-\infty}^{\infty} f(y)(2\pi\delta(y-x))\,\mathrm{d}y \\
&= \int_{y=-\infty}^{\infty} f(y)\delta(y-x)\,\mathrm{d}y \\
&= f(x) \qquad \text{[by the sifting property]}.
\end{aligned}$$

Note that we need to be able to change the order of integration along the way. In the spirit of this course, we do not go into the technical conditions necessary for this to be allowed. ■

We can deduce the Laplace Inversion Theorem from the Fourier Inversion Theorem as follows:

Corollary 85 ***(Inversion Theorem for Laplace Transform)*** *Let f be a differentiable function on $(0,\infty)$ such that $\bar{f}(p)$ exists for $\operatorname{Re} p > c \geqslant 0$. Then for $x > 0$,*

$$f(x) = \frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}\bar{f}(p)\mathrm{e}^{px}\,\mathrm{d}p \qquad (\sigma > c)\,.$$

Proof. Writing $p = \sigma + \mathrm{i}y$ we have

$$\bar{f}(\sigma+\mathrm{i}y) = \int_0^{\infty} f(x)\mathrm{e}^{-(\sigma+\mathrm{i}y)x}\,\mathrm{d}x = \int_0^{\infty}\left[\mathrm{e}^{-\sigma x}f(x)\right]\mathrm{e}^{-\mathrm{i}yx}\,\mathrm{d}x = \hat{g}(y)$$

where

$$g(x) = \mathrm{e}^{-\sigma x}f(x)\mathbb{1}_{[0,\infty)}(x).$$

If we apply the Inverse Fourier Transform we find for $x > 0$ that

$$\mathrm{e}^{-\sigma x}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(\sigma+\mathrm{i}y)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}y$$

which rearranges to

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \bar{f}(\sigma + \mathrm{i}y) \mathrm{e}^{x(\sigma+\mathrm{i}y)} \,\mathrm{d}(\sigma + \mathrm{i}y) = \frac{1}{2\pi i} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \bar{f}(p) \mathrm{e}^{xp} \,\mathrm{d}p.$$

■

Remark 86 *In applied mathematics, it is common to use the Fourier Transform pair in the form*

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{\mathrm{i}kx} \,\mathrm{d}x, \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s) \mathrm{e}^{-\mathrm{i}kx} \,\mathrm{d}k,$$

with the minus in the exponent of the inverse. Here k is often interpreted as a wavenumber (wavelength $= 2\pi/k$). This is our version of the transform with s swapped with $-s$.

Remark 87 *Note that there is a factor of 2π in the inverse Fourier transform which is not present in the Fourier transform itself. This is a consequence of the Fourier transform, as defined here, not being an isometry with respect to the inner product*

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} \,\mathrm{d}x.$$

Consequently some texts define the Fourier transform and its inverse as

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}sx} \,\mathrm{d}x, \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s) \mathrm{e}^{\mathrm{i}sx} \,\mathrm{d}s.$$

This has a nicer symmetry and both the transform and its inverse are now isometries of the above inner product. The obvious downside to this, working with specific examples, is that the Fourier transforms of common functions now involve an unhelpful $\sqrt{2\pi}$ term.

A result demonstrating the near-isometric nature of the Fourier transform is Parseval's Theorem

Theorem 88 ***(Parseval's Theorem for Fourier Transform)*** *(Off-syllabus) Let f and g be integrable functions. Then*

$$\int_{-\infty}^{\infty} \hat{f}(s) \overline{\hat{g}(s)} \,\mathrm{d}s = 2\pi \int_{-\infty}^{\infty} f(x) \overline{g(x)} \,\mathrm{d}x.$$

Proof. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} \hat{f}(s)\overline{\hat{g}(s)}\,\mathrm{d}s &= \int_{s=-\infty}^{\infty}\left(\int_{x=-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x\right)\left(\int_{y=-\infty}^{\infty} \overline{g(y)\mathrm{e}^{-\mathrm{i}sy}}\,\mathrm{d}y\right)\mathrm{d}s \\
&= \int_{s=-\infty}^{\infty}\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f(x)\overline{g(y)}\mathrm{e}^{\mathrm{i}s(y-x)}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f(x)\overline{g(y)}\left(\int_{s=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}s(y-x)}\,\mathrm{d}s\right)\mathrm{d}y\,\mathrm{d}x \\
&= 2\pi\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f(x)\overline{g(y)}\delta(y-x)\,\mathrm{d}y\,\mathrm{d}x \\
&= 2\pi\int_{x=-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x \qquad \text{[by the sifting property].}
\end{aligned}$$

■

Remark 89 *If we take $f = g$, Parseval's theorem shows that 'the energy in the function and the energy in its Fourier transform are the same'. The theorem requires f and g to be square-integrable (i.e. f^2 and g^2 must be integrable). For Lebesgue integration this says that $f \in L^2$, and it is a nice property of the Fourier transform that if $f \in L^2$ then $\hat{f} \in L^2$ too. The corresponding result for functions defined as the sum of a series of basis functions, for example a Fourier series, is made more complicated by the question of completeness (can every function in L^2 be represented as such a sum?).*

Example 90 *Use the Inversion Theorem to determine*

$$\text{(i)} \quad \int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x, \qquad \text{(ii)} \quad \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,\mathrm{d}x, \qquad \text{(iii)} \quad \int_{-\infty}^{\infty} \frac{\cos ax\,\mathrm{d}x}{(x^2+1)^2}$$

where $a > 0$.

Solution. (i) In Example 74 we showed that for $f = \mathbb{1}_{[-1,1]}$ we have

$$\hat{f}(s) = \frac{2\sin s}{s}.$$

Hence by the Inversion Theorem

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin s}{s}\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}s = f(x).$$

If we set $x = 0$ we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin s}{s}\,\mathrm{d}s = f(0) = 1 \qquad \Longrightarrow \qquad \int_{-\infty}^{\infty}\frac{\sin s}{s}\,\mathrm{d}s = \pi.$$

(ii) Similarly from Example 75 for $g(x) = \mathrm{e}^{-a|x|}$ we have

$$\hat{g}(s) = \frac{2a}{a^2+s^2}.$$

Hence by the Inversion Theorem

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2a}{a^2+s^2}\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}s = \mathrm{e}^{-a|x|}.$$

If we set $x = 1$ and take real parts then we find

$$\int_{-\infty}^{\infty}\frac{\cos s}{a^2+s^2}\,\mathrm{d}s = \frac{\pi}{a\mathrm{e}^a}.$$

(iii) We determined in Example 83 the convolution of $h(x) = \mathrm{e}^{-|x|}$ with itself to get

$$(h*h)(x) = (1+|x|)\mathrm{e}^{-|x|}.$$

We know that the Fourier transform of $h(x)$ is $2/(1+s^2)$ and hence by the Inversion Theorem we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}s}{(s^2+1)^2} = (h*h)(x).$$

Taking real parts, setting $x = a$, and renaming our dummy variable, we have

$$\int_{-\infty}^{\infty}\frac{\cos ax\,\mathrm{d}x}{(x^2+1)^2} = 2\pi\frac{h*h}{4}(a) = \frac{\pi}{2}(a+1)\mathrm{e}^{-a}.$$

■

Remark 91 *Note – for later – that replacing s with $-s$ in*

$$\int_{-\infty}^{\infty}\mathrm{e}^{-a|x|}\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x = \frac{2a}{a^2+s^2}$$

means

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-a|x|}\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}x = \frac{a}{\pi(a^2+s^2)}$$

so that the Fourier inverse of $\mathrm{e}^{-a|x|}$ *is*

$$\frac{a}{\pi(a^2+s^2)}.$$

Example 92 *(Convolution of Two Gaussians) Let*

$$f(x) = \frac{1}{a\sqrt{2\pi}}\mathrm{e}^{-x^2/2a^2}, \qquad g(x) = \frac{1}{b\sqrt{2\pi}}\mathrm{e}^{-x^2/2b^2}.$$

We have seen a result which easily shows that

$$\hat{f}(s) = \mathrm{e}^{-a^2s^2/2}, \qquad \hat{g}(s) = \mathrm{e}^{-b^2s^2/2}.$$

Hence

$$\widehat{f * g}(s) = \mathrm{e}^{-a^2s^2/2}\mathrm{e}^{-b^2s^2/2} = \mathrm{e}^{-(a^2+b^2)s^2/2}.$$

Hence the convolution of the Gaussian pdfs for $N(0, a^2)$ and $N(0, b^2)$ is that of $N(0, a^2 + b^2)$.

Chapter 5

Applications to PDEs

Theorem 93 ***(Poisson's Solution to the Dirichlet Problem in the Half-plane)*** *The function $u(x,y)$ satisfies Laplace's equation*

$$u_{xx}(x,y) + u_{yy}(x,y) = 0 \qquad (x \in \mathbb{R}, y > 0)$$

and satisfies the boundary conditions

$$u(x,0) = f(x), \qquad \text{where } f \text{ is integrable,}$$
$$u(x,y) \text{ remains bounded as } x^2 + y^2 \to \infty.$$

Then

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(s)\,\mathrm{d}s}{y^2 + (x-s)^2}$$

Proof. Applying the Fourier transform in the x-variable to the PDE we find that

$$\hat{u}_{yy}(s,y) + (\mathrm{i}s)^2 \hat{u}(s,y) = 0, \qquad \Longrightarrow \qquad \hat{u}_{yy}(s,y) = s^2 \hat{u}(s,y).$$

Solving this we find

$$\hat{u}(s,y) = A(s)\mathrm{e}^{ys} + B(s)\mathrm{e}^{-ys}.$$

By an appropriate choice of functions $\alpha(s)$ and $\beta(s)$ we can rewrite this instead as

$$\hat{u}(s,y) = \alpha(s)\mathrm{e}^{y|s|} + \beta(s)\mathrm{e}^{-y|s|}.$$

Now $\hat{u}(s,y)$ remains bounded as $y \to \infty$ for fixed s and hence we have $\alpha(s) = 0$ and so

$$\hat{u}(s,y) = \beta(s)\mathrm{e}^{-y|s|}.$$

Applying the Fourier transform to the boundary condition we have that

$$\hat{u}(s,0) = \hat{f}(s),$$

and hence $\beta(s) = \hat{f}(s)$ and

$$\hat{u}(s,y) = \hat{f}(s)\mathrm{e}^{-y|s|}.$$

We can write the Fourier inverse of this as a convolution provided we can invert $\mathrm{e}^{-y|s|}$, but we found precisely this inverse in Remark 91 to be

$$\frac{y}{\pi(y^2+x^2)}.$$

Hence the result follows by the convolution theorem. ■

Example 94 *Solve the PDE*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \mathrm{e}^{x-y}$$

subject to the boundary conditions

$$u(x,0) = u(0,y) = 0 \qquad x, y \geqslant 0.$$

Solution. Applying the Laplace transform in the x-direction we find

$$p\bar{u}(p,y) - u(0,y) + \bar{u}_y(p,y) = \frac{\mathrm{e}^{-y}}{p-1},$$

which rearranges to

$$\bar{u}_y(p,y) + p\bar{u}(p,y) = \frac{\mathrm{e}^{-y}}{p-1}.$$

This differential equation has general solution

$$\bar{u}(p,y) = A(p)\mathrm{e}^{-py} + \frac{\mathrm{e}^{-y}}{(p-1)^2}$$

Applying the Laplace transform to the other boundary condition we see that $\bar{u}(p,0) = 0$ and hence

$$A(p) = \frac{-1}{(p-1)^2},$$

so that

$$\bar{u}(p,y) = \frac{\mathrm{e}^{-y} - \mathrm{e}^{-py}}{(p-1)^2}.$$

Recall that the Laplace transform of xe^x is $(p-1)^{-2}$ and hence, inverting, we have

$$\begin{aligned} u(x,y) &= x\mathrm{e}^x\mathrm{e}^{-y} - (x-y)\mathrm{e}^{x-y}H(x-y) \\ &= \begin{cases} x\mathrm{e}^{x-y} & x \leqslant y \\ x\mathrm{e}^{x-y} - (x-y)\mathrm{e}^{x-y} & x \geqslant y \end{cases} \\ &= \begin{cases} x\mathrm{e}^{x-y} & x \leqslant y, \\ y\mathrm{e}^{x-y} & x \geqslant y. \end{cases} \end{aligned}$$

■

Example 95 *Solve the following heat equation using the Laplace transform.*

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2} \qquad 0 \leqslant x \leqslant \pi,\ t \geqslant 0$$

subject to the boundary and initial conditions

$$u(0,t) = 0 = u(\pi,t), \qquad u(x,0) = \sin x, \qquad 0 \leqslant x \leqslant \pi,\ t \geqslant 0.$$

Solution. Applying the Laplace transform to the PDE in the t variable we find

$$p\bar{u}(x,p) - u(x,0) = \bar{u}_{xx}(x,p)$$

which rearranges to

$$\bar{u}_{xx}(x,p) - p\bar{u}(x,p) = -\sin x.$$

This has general solution

$$\bar{u}(x,p) = A(p)\mathrm{e}^{\sqrt{p}x} + B(p)\mathrm{e}^{-\sqrt{p}x} + \frac{\sin x}{p+1}.$$

Applying the Laplace transform to the boundary conditions we get

$$\bar{u}(0,p) = 0 = \bar{u}(\pi,p)$$

and so

$$A(p) + B(p) = 0; \qquad A(p)\mathrm{e}^{\pi\sqrt{p}} + B(p)\mathrm{e}^{-\pi\sqrt{p}} = 0.$$

Solving we find $A(p) = 0 = B(p)$ and hence

$$\bar{u}(x,p) = \frac{\sin x}{p+1},$$

so that, inverting, we see

$$u(x,t) = \mathrm{e}^{-t}\sin x.$$

■

(This is, of course, the usual separation-of-variables solution.)

Chapter 6

The bigger picture, and a forward look

This course is a bit like Clapham Junction: you can get from it to (or pass through it en route to) a vast array of destinations. In this (non-examinable!) conclusion, we take a look at the broader context and see some possible destinations.

6.0.1 More on distributions

More dimensions. It is relatively straightforward to extend the definitions of test functions and distributions to more than one dimension (given a theory of integration). For example, this lets us properly describe a point charge/mass/heat-source in three dimensions using the three-dimensional delta function $\delta(\mathbf{x})$. Then, the steady temperature field generated by a point source of heat of strength Q at the origin, which we know is the radially symmetric solution $T(\mathbf{x}) = Q/(4\pi k|\mathbf{x}|)$ for $|\mathbf{x}| > 0$, satisfies

$$-k\nabla^2 T = Q\delta(\mathbf{x})$$

on *all* of $\mathbb{R}^3$; the line sources, vortices and dipoles of fluid in A10 (Fluids and Waves) satisfy similar equations. The Green's function (B5.2, Applied PDEs) $G(\mathbf{x}, \xi)$ for Laplace's equation in a domain D satisfies (as a function of $\mathbf{x}$) the equation $-\nabla^2 G = \delta(\mathbf{x} - \xi)$ on all of D. This idea is itself an extension of the one-dimensional Green's functions, satisfying $\mathcal{L}y = \delta(x - \xi)$, covered in A6 (DEs 2). All these ideas are unified at the modelling level by thinking of the Green's function as the response of the system to a point influence.

Test functions and weak solutions. We saw earlier the key technical step of moving from a pointwise definition of a function to an averaged definition via an integral. Because the latter is more forgiving, it lets us define, for example, the derivative of a distribution by the integration-by-parts formula $\langle F', \phi\rangle = -\langle F, \phi'\rangle$: the point is that we transfer any possible source of trouble (remember differentiation makes functions less well-behaved) from F to the test function, where it can do no harm as ϕ is smooth. This idea underpins much of the modern analysis of PDEs, via the notion of what are called weak solutions, and you can explore it in B4.3 (Distribution Theory and Fourier Analysis), as well as from a more modelling perspective in B5.2 (Applied PDEs) and B5.4 (Waves & Compressible Flow), where it helps to analyse shock waves such as sonic booms or the Severn Bore.

Other aspects of distributions Another extension of the basic theory is to *pseudofunctions*. This enables us to treat functions such as $1/x$ or $\log x$ on all of $\mathbb{R}$, without worrying about their singularities. The key definition is that of the pseudofunction $1/x$ by its action on a test function ϕ:

$$\langle 1/x, \phi \rangle = \lim_{\epsilon \downarrow 0} \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \frac{\phi(x)\,\mathrm{d}x}{x}.$$

The singularity in the integrand at $x = 0$ is eliminated by the symmetric way we let the interval $(-\epsilon, \epsilon)$ tend to zero. The result is called a Cauchy Principal Value integral and it plays a big part in some applications of complex analysis (covered in C5.6 Applied Complex variables). Its similarity to the Cauchy kernel that you see in Cauchy's Integral Formula is no coincidence. One then defines the derivative of a pseudofunction by its action on a test function in the same way as for a distribution, and you might like to use this idea to show that the ordinary (integrable) function $\log|x|$ and the pseudofunction $1/x$ satisfy $\mathrm{d}\log|x|/\mathrm{d}x = 1/x$ on all of $\mathbb{R}$. (It is a short extension to define $1/x^2$ as the derivative of $-1/x$, and then you can prove the amusing (but correct) formula $\langle 1/x^2, 1\rangle = 0$ — to see why it is amusing, write it as an 'integral'.

If you are interested in probability, B8.1 (Probability, Measure & Martingales) starts with an outline of Measure Theory, in which the delta measure (our delta function) allows us to bring together discrete and continuous random variables in a single setting.

6.0.2 Fourier Transforms and distributions

Earlier in these notes we were happy to take the Fourier Transform of (for example) the delta function, to get $\hat{\delta} = 1$. Strictly speaking, some more test-function machinery is needed to make this work. It turns out that we need to modify our definition of test functions slightly, to allow them to be nonzero as $|x| \to \infty$, provided they decay fast enough in the limit. They then have the nice property that the Fourier Transform of a test function is also a test function (not true for the previous kind). The result is the space of 'tempered distributions', and the formal definition of the Fourier Transform of a tempered distribution F is via a test function $\phi(x)$:

$$\langle \hat{F}, \phi \rangle = \langle F, \hat{\phi} \rangle$$

with inverse

$$\langle \check{F}, \phi \rangle = \langle F, \check{\phi} \rangle.$$

For all practical purposes, these distributions are the same as our earlier ones.

6.0.3 Where do transforms come from?

A first answer to this question is that they are 'Fourier series on an infinite interval'. To see this very informally, consider a function $f(x)$ defined on the interval $(-L, L)$. We may write $f(x)$ as the complex Fourier series, with coefficients f_n,

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \mathrm{e}^{\mathrm{i}n\pi x/L}, \quad \text{where} \quad f_n = \frac{2}{L}\int_{-L}^{L} f(x')\mathrm{e}^{-\mathrm{i}n\pi x'/L}\mathrm{d}x',$$

from which

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \mathrm{e}^{\mathrm{i}n\pi x/L} = \sum_{n=-\infty}^{\infty} \left(\frac{2}{L}\int_{-L}^{L} f(x')\mathrm{e}^{-\mathrm{i}n\pi x'/L}\mathrm{d}x' \right) \mathrm{e}^{\mathrm{i}n\pi x/L}.$$

Now compare this with the Fourier Transform pair when $f(x)$ is defined on all of $\mathbb{R}$:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)\mathrm{e}^{\mathrm{i}sx}\mathrm{d}s = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')\mathrm{e}^{-\mathrm{i}sx'}\mathrm{d}x' \right) \mathrm{e}^{\mathrm{i}sx}\mathrm{d}s.$$

If we let $L \to \infty$, the two expressions agree if we interpret the sum as a Riemann integral by setting $n\pi/L = s$ and the increment $\pi/L = \mathrm{d}s$. Although not rigorous, this is certainly a strong clue that the Fourier Transform is indeed an extension of Fourier series. Unfortunately, no such easy idea is available for Laplace Transforms, and so we must look deeper. Those taking A6 (DEs 2) will recognise some of what follows.

Recall from linear algebra that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ can be represented by a matrix $\mathbf{A}$ with respect to any basis. When $\mathbf{A}$ is symmetric, we know that the eigenvalues are all real and the eigenvectors are orthogonal. When $\mathbf{A}$ also has rank n (so no eigenvalue vanishes), the (normalised) eigenvectors $\mathbf{v}_i$ form an orthonormal basis which is 'natural' for this transformation. Any other vector $\mathbf{w}$ has an expansion $\mathbf{w} = \sum_i c_i \mathbf{v}_i$ where the coefficients take the simple form $c_i = \langle \mathbf{v}_i, \mathbf{w} \rangle$ (here $\langle \cdot\,, \cdot \rangle$ is the usual inner product). In particular, the solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \sum_i \langle \mathbf{v}_i, \mathbf{b} \rangle \mathbf{v}_i / \lambda_i$.

Now recall Fourier series. Any continuous function $f(x)$ which vanishes at $x = \pm L$ has the Fourier series representation

$$f(x) = \sum_n \Big(b_n \sin(n\pi x/L) + a_n \cos\left((2n+1)\pi x/2L\right) \Big),$$

where each coefficient is given by integrating $f(x)$ against the corresponding basis function. These basis functions all satisfy the 'eigenproblem'

$$-\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \lambda y, \quad -L < x < L, \qquad y(\pm L) = 0,$$

where λ, the eigenvalue, is either $(n\pi/L)^2$ or $((2n+1)\pi/2L)^2$. So this differential operator ($-\mathrm{d}^2/\mathrm{d}x^2$ plus boundary conditions) leads to a natural basis for the representation of the solution to $-\mathrm{d}^2y/\mathrm{d}x^2 = g(x)$ with $y(\pm L) = 0$. More practically, it is why separation of variables works for the one-dimensional heat and wave equations, as the 'spatial' differential operator is precisely '$-\mathrm{d}^2/\mathrm{d}x^2$ plus boundary conditions'. If we have a different operator, we may get different basis functions; for example, separation of variables for a radially symmetric solution of the two-dimensional heat equation leads to a series in terms of Bessel's function of order zero, J_0.

It is a small step to see that this idea can apply to transforms as well. The Fourier Transform arises from the (self-adjoint) eigenproblem

$$-\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \lambda y, \quad -\infty < x < \infty, \qquad y \text{ bounded as } x \to \pm\infty,$$

for which $\lambda = s^2$ is real and positive. Note that the discrete spectrum (countable number of eigenvalues) we saw on a finite interval has become a continuous one as s can take any real value. Likewise the Laplace transform arises from the (not self-adjoint) eigenproblem

$$-\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda y, \quad 0 < x < \infty, \qquad y \text{ bounded as } x \to \infty,$$

for which $\lambda = p$. Other differential operators give other transforms; for example, the Bessel operator leads to the Hankel Transform, and so on. For more on the theoretical underpinnings of these calculations, B4.1 and B4.2 Functional Analysis are the courses to take.

6.0.4 Further uses of transforms

Many areas of applied mathematics use transforms of one kind or another: you can expect to see them whenever the underlying model is a linear ordinary or partial differential equation. Prominent examples include wave motion in fluid mechanics (sound waves in a compressible fluid, water waves; A10), solid mechanics (the elastic equivalent of sound waves; C5.2) and electromagnetism (B7.2). This last gives a good example of the power of the Laplace Transform: whereas most of the examples we have seen in this course are straightforward and could have been solved just as easily by other methods, the boot is on the other foot when it comes to large systems of ODEs, such as arise in the theory of linear electrical circuits, for example models of power grids subject to unexpected shocks (sorry ...). Here a huge system of ODEs modelling the interaction of the inductances, capacitances and resistances of the system is reduced by the Laplace Transform to a much more basic problem in linear algebra.

On a much smaller scale, the Fourier Transform is important in quantum mechanics: Heisenberg's Uncertainty Principle follows from the Fourier Transform result

$$E_x E_s \geq \frac{1}{4}, \qquad \text{where} \qquad E_x = \frac{\int_{-\infty}^{\infty} x^2 \left(f(x)\right)^2 \, \mathrm{d}x}{\int_{-\infty}^{\infty} \left(f(x)\right)^2 \, \mathrm{d}x} \quad \text{and} \quad E_s = \frac{\int_{-\infty}^{\infty} s^2 |\hat{f}(s)|^2 \, \mathrm{d}s}{\int_{-\infty}^{\infty} |\hat{f}(s)|^2 \, \mathrm{d}s}.$$

In probability, the characteristic function of a random variable is essentially the Fourier Transform of its density. Signals processing is another fertile area of use for the Fourier transform, with the recent detection of gravitational waves by the LIGO experiment being just one example of its use. Here the independent variable x becomes time, t, and then we think of the transform variable s as an angular frequency, often written as ω; sometimes a factor 2π appears in the exponent, indicating that frequencies are measured in Hz.[1] The transform then takes a signal in the 'time domain' into the 'frequency domain' and the FT of a signal represents the amplitudes in its decomposition into (a continuum of) frequencies; the numerical computation of the transform is often done using the celebrated Fast Fourier Transform (FFT). In this context, the result that $\hat{\delta} = 1$ says that the delta function (in time) is a bang which contains an equal amount of every frequency (a fact which finds application in seismology). Put another way, a signal that is completely localised in time is uniformly spread out in frequency, an extreme case of the uncertainty principle mentioned above. This motivates the idea of wavelets, which are functions localised both in time and frequency, and which are used (among other things) to generate hierarchical compression of digital images. A final example is the famous Radon transform which is closely related to the Fourier Transform and underpins the imaging of a patient having a CT scan.

Finally, note that oten the solution of a transform problem comes in the form of an inversion integral which cannot be calculated explicitly, and then a systematic approximation may be useful (C5.5 Perturbation Methods).

[1] And, in engineering, $\mathrm{i}^2 = -1$ becomes $\mathrm{j}^2 = -1 \ldots$