# Week 2 — Techniques of Integration

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**Abstract**

Integration by Parts. Substitution. Rational Functions. Partial Fractions. Trigonometric Substitutions. Numerical Methods.

**Remark 1** *We will demonstrate each of the techniques here by way of examples, but concentrating each time on what general aspects are present. Integration, though, is not something that should be learnt as a table of formulae, for at least two reasons: one is that most of the formula would be far from memorable, and the second is that each technique is more flexible and general than any memorised formula ever could be. If you can approach an integral with a range of techniques at hand you will find the subject less confusing and not be fazed by new and different functions.*

**Remark 2** *When it comes to checking your answer there are various quick rules you can apply. If you have been asked to calculate an indefinite integral then, if it's not too complicated, you can always differentiate your answer to see if you get the original integrand back. This, of course, applies to definite integrals as well before you enter the integral's limits. Even at this point, you can still apply some simple estimation rules: if your integrand is positive (or negative) then so should your answer be; if your integrand is less than a well-known function, then its integral will be less than the integral of the well-known function. These can be useful checks to quickly apply at the end of the calculation.*

## 1 Integration by Parts

Integration by parts (IBP) can be used to tackle products of functions, but not just any product. Suppose we have an integral

$$\int f(x)\, g(x)\ \mathrm{d}x$$

in mind. This will be susceptible to IBP if one of these functions integrates, or differentiates, perhaps repeatedly, to something simpler, whilst the other function differentiates and integrates to something of the same kind. Typically then $f(x)$ might be a polynomial which, after differentiating enough times, will become a constant; $g(x)$ on the other hand could be something like $e^x$, $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, all of which are functions which continually integrate to something similar. This remark reflects the nature of the formula for IBP which is:

**Proposition 3** *(Integration by Parts) Let $F$ and $G$ be functions with derivatives $f$ and $g$. Then*

$$\int F(x)\, g(x)\ dx = F(x)\, G(x) - \int f(x)\, G(x)\ dx.$$

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IBP takes the integral of a product and leaves us with another integral of a product — but as we commented above, the point is that $f(x)$ should be a simpler function than $F(x)$ was whilst $G(x)$ should be no worse a function than $g(x)$ was.

**Proof.** The proof is simple — we just integrate the product rule of differentiation below, and rearrange.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F(x)\,G(x)\right)=F(x)\,g(x)+f(x)\,G(x)$$

■

**Example 4** *Determine*

$$\int x^2\sin x\ dx \quad and \quad \int_0^1 x^3e^{2x}\ dx.$$

Clearly $x^2$ will be the function that we need to differentiate down, and $\sin x$ is the function that will integrate *in house.* So we have, with *two* applications of IBP:

$$\begin{aligned}
\int x^2\sin x\ \mathrm{d}x &= x^2(-\cos x)-\int 2x(-\cos x)\ \mathrm{d}x \quad \text{[IBP]}\\
&= -x^2\cos x+\int 2x\cos x\ \mathrm{d}x \quad \text{[Rearranging]}\\
&= -x^2\cos x+2x\sin x-\int 2\sin x\ \mathrm{d}x \quad \text{[IBP]}\\
&= -x^2\cos x+2x\sin x-2(-\cos x)+\text{const.}\\
&= \left(2-x^2\right)\cos x+2x\sin x+\text{const.} \quad \text{[Rearranging]}
\end{aligned}$$

In a similar fashion

$$\begin{aligned}
\int_0^1 x^3e^{2x}\ \mathrm{d}x &= \left[x^3\frac{e^{2x}}{2}\right]_0^1-\int_0^1 3x^2\frac{e^{2x}}{2}\ \mathrm{d}x \quad \text{[IBP]}\\
&= \frac{e^2}{2}-\left(\left[3x^2\frac{e^{2x}}{4}\right]_0^1-\int_0^1 6x\frac{e^{2x}}{4}\ \mathrm{d}x\right) \quad \text{[IBP]}\\
&= \frac{e^2}{2}-\frac{3e^2}{4}+\left[6x\frac{e^{2x}}{8}\right]_0^1-\int_0^1 6\frac{e^{2x}}{8}\ \mathrm{d}x \quad \text{[IBP]}\\
&= \frac{-e^2}{4}+\frac{3e^2}{4}-\left[\frac{6e^{2x}}{16}\right]_0^1\\
&= \frac{e^2}{8}+\frac{3}{8}.
\end{aligned}$$

This is by far the main use of IBP, the idea of eventually differentiating out one of the two functions. There are other important uses of IBP which don't quite fit into this type. These next two examples fall into the original class, but are a little unusual : in these cases we choose to integrate the polynomial factor instead as it is easier to differentiate the other factor. This is the case when we have a logarithm or an inverse trigonometric function as the second factor.

**Example 5** *Evaluate*

$$\int (2x-1)\ln\left(x^2+1\right)\ dx \quad and \quad \int \left(x^2-4\right)\tan^{-1}x\ dx.$$

In both cases integrating the second factor looks rather daunting but each factor differentiates nicely; recall that

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x=\frac{1}{x} \quad \text{and that} \quad \frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x=\frac{1}{1+x^2}.$$

So if we apply IBP to the above examples then we get

$$\int (2x-1)\ln\left(x^2+1\right)\ \mathrm{d}x = \left(x^2-x\right)\ln\left(x^2+1\right) - \int \left(x^2-x\right)\frac{2x}{x^2+1}\ \mathrm{d}x,$$

and

$$\int \left(3x^2-4\right)\tan^{-1}x\ \mathrm{d}x = \left(x^3-4x\right)\tan^{-1}x - \int \left(x^3-4x\right)\frac{1}{x^2+1}\ \mathrm{d}x.$$

Here we will stop for the moment — we will see how to determine these integrals, the integrands of which are known as rational functions, in section 3.

In the same vein as this we can use IBP to integrate functions which, at first glance, don't seem to be a product — this is done by treating a function $F(x)$ as the product $F(x)\times 1$.

**Example 6** *Evaluate*

$$\int \ln x\ dx \text{ and } \int \tan^{-1}x\ dx.$$

With IBP we see (integrating the 1 and differentiating the $\ln x$)

$$\begin{aligned}
\int \ln x\ \mathrm{d}x &= \int 1\times\ln x\ \mathrm{d}x \\
&= x\ln x - \int x\frac{1}{x}\ \mathrm{d}x \\
&= x\ln x - \int \mathrm{d}x \\
&= x\ln x - x + \text{const.}
\end{aligned}$$

and similarly

$$\begin{aligned}
\int \tan^{-1}x\ \mathrm{d}x &= \int 1\times\tan^{-1}x\ \mathrm{d}x \\
&= x\tan^{-1}x - \int x\frac{1}{1+x^2}\ \mathrm{d}x \\
&= x\tan^{-1}x - \frac{1}{2}\ln\left(1+x^2\right) + \text{const.}
\end{aligned}$$

spotting this by *inspection* or by using *substitution* (see the next section).

Sometimes both functions remain *in house,* but we eventually return to our original integrand.

**Example 7** *Determine*

$$\int e^x\sin x\ dx.$$

Both of these functions now remain in house, but if we apply IBP twice, integrating the $e^x$ and differentiating the $\sin x$, then we see

$$\begin{aligned}
\int e^x\sin x\ \mathrm{d}x &= e^x\sin x - \int e^x\cos x\ \mathrm{d}x \quad \text{[IBP]} \\
&= e^x\sin x - \left(e^x\cos x - \int e^x(-\sin x)\ \mathrm{d}x\right) \\
&= e^x(\sin x - \cos x) - \int e^x\sin x\ \mathrm{d}x.
\end{aligned}$$

We see that we have returned to our original integral, and so we can rearrange this equality to get

$$\int e^x\sin x\ \mathrm{d}x = \frac{1}{2}e^x(\sin x-\cos x) + \text{const.}$$

## 2 Substitution

In many ways the hardest aspect of integration to teach, a technique that can become almost an art form, is substitution. Substition is such a varied and flexible approach that it is impossible to classify (and hence limit) its uses, and quite difficult even to find general themes within. We shall discuss later some standard trigonometric substitutions useful in integrating rational functions. For now we will simply state what substitution involves and highlight one difficulty than can occur (and cause errors) unless substitution is done carefully.

**Proposition 8** *Let $g : [c,d] \to [a,b]$ be an increasing function, such that $g(c) = a$ and $g(d) = b$, and which has derivative $g'$. Then*

$$\int_a^b f(x)\ dx = \int_c^d f(g(t))\, g'(t)\ dt.$$

*Similarly if $g : [c,d] \to [a,b]$ is a decreasing function, such that $g(c) = b$ and $g(d) = a$, then*

$$\int_a^b f(x)\ dx = \int_d^c f(g(t))\, g'(t)\ dt.$$

The important point here is that the function $g$ be increasing or decreasing so that it is a *bijection* from $[c,d]$ to $[a,b]$ — what this technical term simply means is that to each value of $x$ in the range $[a,b]$ there should be *exactly one* value of $t$ in the range $[c,d]$ such that $g(t) = x$. Here is an example of what might go wrong if substitution is incorrectly applied.

**Example 9** *Evaluate*

$$\int_{-1}^2 x^2\ dx.$$

This is not a difficult integral and we would typically not think of using substitution to do this; we would just proceed and find

$$\int_{-1}^2 x^2\ \mathrm{d}x = \left[\frac{x^3}{3}\right]_{-1}^2 = \frac{1}{3}\left(2^3 - (-1)^3\right) = \frac{9}{3} = 3.$$

But suppose that we'd tried to use (in a less than rigorous fashion) the substitution $u = x^2$ here. We'd see that

$$\begin{aligned} \mathrm{d}u &= 2x\,\mathrm{d}x = 2\sqrt{u}\,\mathrm{d}x \text{ so that } \mathrm{d}x = \frac{\mathrm{d}u}{2\sqrt{u}} \\ \text{and when } x &= -1, u = 1 \text{ and when } x = 2,\ u = 4. \end{aligned}$$

So surely we'd find

$$\int_{-1}^2 x^2\ \mathrm{d}x = \int_1^4 u\frac{\mathrm{d}u}{2\sqrt{u}} = \frac{1}{2}\int_1^4 \sqrt{u}\ \mathrm{d}u = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^4 = \frac{1}{3}(8-1) = \frac{7}{3}.$$

What's gone wrong is that the assignment $u = x^2$ doesn't provide a bijection between $[-1,2]$ and $[1,4]$ as the values in $[-1,0]$ square to the same values as those in $[0,1]$. The missing $2/3$ error in the answer is in fact the integral $\int_{-1}^1 x^2\ dx$. If we'd particularly wished to use this substitution here then it could have been correctly made by splitting our integral as

$$\int_{-1}^2 x^2\ \mathrm{d}x = \int_{-1}^0 x^2\ \mathrm{d}x + \int_0^2 x^2\ \mathrm{d}x$$

and using the substitution $u = x^2$ separately on each integral; this would work because $u = x^2$ gives a bijection between $[-1,0]$ and $[0,1]$, and between $[0,2]$ and $[0,4]$.

Here are some examples where substitution can be applied, provided some care is taken.

**Example 10** *Evaluate the following integrals*

$$\int_0^1 \frac{1}{1+e^x}\ dx, \qquad \int_{-\pi}^{\pi} \frac{\sin x}{1+\cos x}\ dx.$$

In the first integral a substitution that might suggest itself is $u = 1 + e^x$ or $u = e^x$; let's try the first of these $u = 1 + e^x$. As $x$ varies from $x = 0$ to $x = 1$ then $u$ varies from $u = 2$ to $u = 1 + e$. Morever $u$ is increasing with $x$ so that the rule $u = 1 + e^x$ is a bijection from the $x$-values in $[0, 1]$ to the $u$-values in the range $[2, 1 + e]$. We also have that

$$\mathrm{d}u = e^x \mathrm{d}x = (u-1)\,\mathrm{d}x.$$

So

$$\begin{aligned}
\int_0^1 \frac{1}{1+e^x}\,\mathrm{d}x &= \int_2^{1+e} \frac{1}{u}\frac{\mathrm{d}u}{u-1} \quad \text{[substitution]} \\
&= \int_2^{1+e} \left(\frac{1}{u-1} - \frac{1}{u}\right) \mathrm{d}u \text{ [using partial fractions]} \\
&= \left[\ln|u-1| - \ln|u|\right]_2^{1+e} \\
&= \ln(e) - \ln(1+e) - \ln 1 + \ln 2 \\
&= 1 + \ln\left(\frac{2}{1+e}\right).
\end{aligned}$$

For the second integral, it would seem sensible to use $u = 2 + \cos x$ or $u = \cos x$ here. Let's try the second one: $u = \cos x$. Firstly note that $u$ is not a bijection on the range $[-\pi/2, \pi]$, it takes the same values in the range $[-\pi/2, 0]$ as it does in the range $[0, \pi/2]$. In fact the integrand is *odd* (that is $f(-x) = -f(x)$) and so its integral between $x = -\pi/2$ and $\pi/2$ will be zero automatically. So we can write

$$\begin{aligned}
\int_{-\pi/2}^{\pi} \frac{\sin x}{2+\cos x}\,\mathrm{d}x &= \int_{-\pi/2}^{\pi/2} \frac{\sin x}{2+\cos x}\,\mathrm{d}x + \int_{\pi/2}^{\pi} \frac{\sin x}{2+\cos x}\,\mathrm{d}x \\
&= \int_{\pi/2}^{\pi} \frac{\sin x}{2+\cos x}\,\mathrm{d}x.
\end{aligned}$$

Now we can use the substitution $u = \cos x$ noticing that $\mathrm{d}u = -\sin x\ \mathrm{d}x$, when $x = \pi/2, u = 0$ and when $x = \pi, u = -1$, so that

$$\begin{aligned}
\int_{-\pi/2}^{\pi} \frac{\sin x}{2+\cos x}\,\mathrm{d}x &= \int_0^{-1} \frac{-\mathrm{d}u}{2+u} \\
&= -\left[\ln|2+u|\right]_0^{-1} \\
&= -(\ln 1 - \ln 2) \\
&= \ln 2.
\end{aligned}$$

# 3 Rational Functions

A *rational function* is one of the form

$$\frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0},$$

where the $a_i$ and $b_i$ are constants — that is, the quotient of two polynomials. In principle, (because of the Fundamental Theorem of Algebra which says that the roots of the denominator can all be found in the complex numbers), it is possible to rewrite the denominator as

$$b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0 = p_1(x)\, p_2(x) \cdots p_k(x)$$

where the polynomials $p_i(x)$ are either linear factors (of the form $Ax + B$) or quadratic factors ($Ax^2 + Bx + C$) with $B^2 < 4AC$ and complex conjugates for roots. From here we can use partials fractions to simplify the function.

## 3.1 Partial Fractions

Given a rational function

$$\frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{p_1(x)\, p_2(x) \cdots p_k(x)}$$

where the factors in the denominator are linear or quadratic terms, we follow several simple steps to put it into a form we can integrate.

1. if the numerator has greater degree than the denominator, then we divide the denominator into the numerator (using polynomial long division) till we have an expression of the form

$$P(x) + \frac{A_i x^i + A_{i-1} x^{i-1} + \cdots + A_0}{p_1(x)\, p_2(x) \cdots p_k(x)}$$

   where $P(x)$ is a polynomial, and the numerator $A_i x^i + A_{i-1} x^{i-1} + \cdots + A_0$ now has a strictly smaller degree than the denominator $p_1(x)\, p_2(x) \cdots p_k(x)$. Of course, integrating the polynomial part $P(x)$ will not cause us any difficulty so we will ignore it from now on.

2. Let's suppose, for now, that none of the factors in the denominator are the same. In this case we can use partial fractions to rewrite this new rational function as

$$\frac{A_i x^i + A_{i-1} x^{i-1} + \cdots + A_0}{p_1(x)\, p_2(x) \cdots p_k(x)} = \frac{\alpha_1(x)}{p_1(x)} + \frac{\alpha_2(x)}{p_2(x)} + \cdots + \frac{\alpha_k(x)}{p_k(x)}$$

   where each polynomial $\alpha_i(x)$ is of smaller degree than $p_i(x)$. This means that we have rewritten the rational function in terms of rational functions of the form

$$\frac{A}{Bx + C} \quad \text{and} \quad \frac{Ax + B}{Cx^2 + Dx + E}$$

   which we will see how to integrate in the next subsection.

3. If however a factor, say $p_1(x)$, is repeated $N$ times say, then rather than the $\alpha_1(x)/p_1(x)$ term in the equation above, the best we can do with partial fractions is to reduce it to an expression of the form

$$\frac{\beta_1(x)}{p_1(x)} + \frac{\beta_2(x)}{(p_1(x))^2} + \cdots + \frac{\beta_N(x)}{(p_1(x))^N}$$

   where the polynomials $\beta_i(x)$ have smaller degree than $p_1(x)$. This means the final expression may include functions of the form

$$\frac{A}{(Bx + C)^n} \quad \text{and} \quad \frac{Ax + B}{(Cx^2 + Dx + E)^n} \quad \text{where } D^2 < 4CE.$$

**Example 11** *Use the method of partial fractions to write the following rational function in simpler form*

$$\frac{x^5}{(x-1)^2(x^2+1)}.$$

The numerator has degree 5 whilst the denominator has degree 4, so we will need to divide the denominator into the numerator first. The denominator expands out to

$$(x-1)^2\left(x^2+1\right) = x^4 - 2x^3 + 2x^2 - 2x + 1.$$

Using polynomial long-division we see that

$$\begin{array}{r|rrrrrr}
 & & & & & x & +2 \\ \cline{2-7}
x^4-2x^3+2x^2-2x+1 & x^5 & +0x^4 & +0x^3 & +0x^2 & +0x & +0 \\
 & x^5 & -2x^4 & +2x^3 & -2x^2 & +x & \\ \cline{2-6}
 & & 2x^4 & -2x^3 & +2x^2 & -x & +0 \\
 & & 2x^4 & -4x^3 & +4x^2 & -4x & +2 \\ \cline{3-7}
 & & & 2x^3 & -2x^2 & +3x & -2
\end{array}$$

So we have that

$$\frac{x^5}{(x-1)^2(x^2+1)} \equiv x + 2 + \frac{2x^3-2x^2+3x-2}{(x-1)^2(x^2+1)},$$

which leaves us to find the constants $A, B, C, D$, in the identity

$$\frac{2x^3-2x^2+3x-2}{(x-1)^2(x^2+1)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}.$$

Multiplying through by the denominator, we find

$$2x^3 - 2x^2 + 3x - 2 \equiv A(x-1)\left(x^2+1\right) + B\left(x^2+1\right) + (Cx+D)(x-1)^2.$$

As this holds for all values of $x$, then we can set $x = 1$ to deduce

$$2 - 2 + 3 - 2 = 1 = 2B \quad \text{and so } B = \frac{1}{2}.$$

If we set $x = 0$ then we also get that

$$-2 = -A + \frac{1}{2} + D. \tag{1}$$

Other things we can do are to compare the coefficients of $x^3$ on either side which gives

$$2 = A + C \tag{2}$$

and to compare the coefficients of $x$ which gives

$$3 = A + C - 2D. \tag{3}$$

Substituting (2) into (3) yields $3 = 2 - 2D$ and so $D = -1/2$. From equation (1) this means that $A = 2$ and so $C = 0$. Finally then we have

$$\frac{2x^3-2x^2+3x-2}{(x-1)^2(x^2+1)} \equiv \frac{2}{x-1} + \frac{1/2}{(x-1)^2} - \frac{1/2}{x^2+1}$$

and

$$\frac{x^5}{(x-1)^2(x^2+1)} \equiv x + 2 + \frac{2}{x-1} + \frac{1/2}{(x-1)^2} - \frac{1/2}{x^2+1}.$$

### 3.2 Trigonometric Substitutions

If we look now at the function we are faced with, namely

$$x+2+\frac{2}{x-1}+\frac{1/2}{(x-1)^2}-\frac{1/2}{x^2+1},$$

then only the final term is something that would cause trouble from an integrating point. To deal with such functions we recall the trigonometric identities

$$\sin^2\theta+\cos^2\theta=1, \quad 1+\tan^2\theta=\sec^2\theta, \quad 1+\cot^2\theta=\csc^2\theta. \tag{4}$$

So a substitution of the form $x=\tan\theta$ into an expression like $1+x^2$ simplifies it to $\sec^2\theta$. Noting

$$\mathrm{d}x=\sec^2\theta\ \mathrm{d}\theta$$

we find

$$\begin{aligned}\int\frac{\mathrm{d}x}{1+x^2} &= \int\frac{\sec^2\theta\ \mathrm{d}\theta}{1+\tan^2\theta}\\ &= \int\frac{\sec^2\theta\ \mathrm{d}\theta}{\sec^2\theta}\\ &= \int\mathrm{d}\theta\\ &= \theta+\text{const.}\\ &= \tan^{-1}x+\text{const.}\end{aligned}$$

So returning to our example we see

$$\begin{aligned}\int\frac{x^5\ \mathrm{d}x}{(x-1)^2(x^2+1)} &\equiv \int\left(x+2+\frac{2}{x-1}+\frac{1/2}{(x-1)^2}-\frac{1/2}{x^2+1}\right)\mathrm{d}x\\ &= \frac{x^2}{2}+2x+2\ln|x-1|-\frac{1/2}{x-1}-\frac{1}{2}\tan^{-1}x+\text{const.}\end{aligned}$$

Returning to the most general form of a rational function, we were able to reduce (using partial fractions) the problem to integrands of the form

$$\frac{A}{(Bx+C)^n} \quad\text{and}\quad \frac{Ax+B}{(Cx^2+Dx+E)^n} \quad\text{where } D^2<4CE.$$

Integrating functions of the first type causes us no difficulty as

$$\int\frac{A\ \mathrm{d}x}{(Bx+C)^n}=\begin{cases}\frac{A}{B(1-n)}(Bx+C)^{1-n}+\text{const.} & n\neq 1;\\ \frac{A}{B}\ln|Bx+C|+\text{const.} & n=1.\end{cases}$$

The second integrand can be simplified, firstly by completing the square and then with a trigonometric substitution. Note that

$$Cx^2+Dx+E=C\left(x+\frac{D}{2C}\right)^2+\left(E-\frac{D^2}{4C}\right).$$

If we make a substitution of the form $u=x+D/2C$ then we can simplify this integral to something of the form

$$\int\frac{(au+b)\ \mathrm{d}u}{(u^2+k^2)^n} \quad\text{for new constants } a,b \text{ and } k>0.$$

Part of this we can integrate directly:

$$\int\frac{u\ \mathrm{d}u}{(u^2+k^2)^n}=\begin{cases}\frac{1}{2(1-n)}\left(u^2+k^2\right)^{1-n}+\text{const.} & n\neq 1;\\ \frac{1}{2}\ln\left(u^2+k^2\right)+\text{const.} & n=1.\end{cases}$$

The other integral

$$\int \frac{\mathrm{d}u}{\left(u^2+k^2\right)^n}$$

can be simplified with a trigonometric substitution $u = k\tan\theta$, the integral becoming

$$\begin{aligned}
\int \frac{\mathrm{d}u}{\left(u^2+k^2\right)^n} &= \int \frac{k\sec^2\theta\ \mathrm{d}\theta}{\left(k^2\tan^2\theta+k^2\right)^n} \\
&= \frac{1}{k^{2n-1}} \int \frac{\sec^2\theta\ \mathrm{d}\theta}{\left(\sec^2\theta\right)^n} \\
&= \frac{1}{k^{2n-1}} \int \cos^{2n-2}\theta\ \mathrm{d}\theta.
\end{aligned}$$

The $n = 0, 1, 2$ cases can all easily be integrated. We will see in the next section on Reduction Formulae how to deal generally with integrals of this form. For now we will simply give an example where $n = 2$.

**Example 12** *Determine*

$$I = \int \frac{dx}{\left(3x^2+2x+1\right)^2}$$

Remember that the first step is to complete the square:

$$\begin{aligned}
I &= \int \frac{\mathrm{d}x}{\left(3x^2+2x+1\right)^2} \\
&= \frac{1}{9} \int \frac{\mathrm{d}x}{\left(x^2+\frac{2}{3}x+\frac{2}{3}\right)^2} \\
&= \frac{1}{9} \int \frac{\mathrm{d}x}{\left(\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right)^2}
\end{aligned}$$

Our first substitution is simply a translation — let $u = x + 1/3$ noting that $\mathrm{d}u = \mathrm{d}x$ :

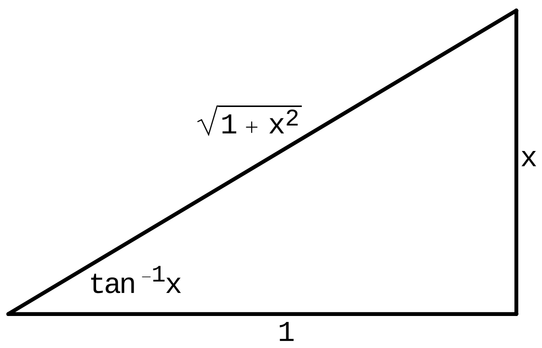
$$I = \frac{1}{9} \int \frac{\mathrm{d}x}{\left(\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right)^2} = \frac{1}{9} \int \frac{\mathrm{d}u}{\left(u^2+2/9\right)^2}$$

Then we set $u = \frac{\sqrt{2}}{3}\tan\theta$ to further simplify the integral. So

$$\begin{aligned}
I &= \frac{1}{9} \int \frac{\left(2/\sqrt{3}\right)\sec^2\theta\ \mathrm{d}\theta}{\left(2\sec^2\theta/9\right)^2} \\
&= \frac{1}{9} \times \frac{2}{\sqrt{3}} \times \left(\frac{9}{2}\right)^2 \int \cos^2\theta\ \mathrm{d}\theta \\
&= \frac{9}{2\sqrt{3}} \int \frac{1}{2}\left(1+\cos 2\theta\right)\ \mathrm{d}\theta \quad [\text{using } \cos 2\theta = 2\cos^2\theta - 1] \\
&= \frac{9}{4\sqrt{3}} \left(\theta + \frac{1}{2}\sin 2\theta\right) + \text{const.} \\
&= \frac{9}{4\sqrt{3}} \left(\theta + \sin\theta\cos\theta\right) + \text{const.} \quad [\text{using } \sin 2\theta = 2\sin\theta\cos\theta] \\
&= \frac{9}{4\sqrt{3}} \left(\tan^{-1}\frac{3u}{\sqrt{2}} + \sin\tan^{-1}\frac{3u}{\sqrt{2}}\cos\tan^{-1}\frac{3u}{\sqrt{2}}\right) + \text{const.}
\end{aligned}$$

by undoing the substitution $u = \frac{\sqrt{2}}{3}\tan\theta$.

From the right-angled triangle




we see that

$$\sin\tan^{-1}x = \frac{x}{\sqrt{1+x^2}} \quad \text{and} \quad \cos\tan^{-1}x = \frac{1}{\sqrt{1+x^2}}.$$

So

$$\begin{aligned} I &= \frac{9}{4\sqrt{3}}\left(\tan^{-1}\frac{3u}{\sqrt{2}} + \frac{3u/\sqrt{2}}{\sqrt{1+9u^2/2}} \times \frac{1}{\sqrt{1+9u^2/2}}\right) + \text{const.} \\ &= \frac{9}{4\sqrt{3}}\left(\tan^{-1}\frac{3u}{\sqrt{2}} + \frac{6u}{\sqrt{2}\,(2+9u^2)}\right) + \text{const.} \\ &= \frac{9}{4\sqrt{3}}\left(\tan^{-1}\left(\frac{3}{\sqrt{2}}\left(x+\frac{1}{3}\right)\right) + \frac{6x+2}{\sqrt{2}\,(9x^2+6x+3)}\right) + \text{const.} \\ &= \frac{9}{4\sqrt{3}}\left(\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + \frac{2(3x+1)}{3\sqrt{2}\,(3x^2+2x+1)}\right) + \text{const.} \end{aligned}$$

This example surely demonstrates the importance of remembering the method and not the formula!

## 3.3 Further Trigonometric Substitutions

The trigonometric identities in equation (4) can be applied in more general cases than those above used for integrating rational functions above. A similar standard trigonometric integral is

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \sin^{-1}x + \text{const.}$$

This can be deduced in exactly the same way: this time we make use of the trigonometric identity

$$1 - \sin^2\theta = \cos^2\theta$$

and make a substitution $x = \sin\theta$ to begin this calculation. Likewise the integral

$$\int \frac{\mathrm{d}x}{\sqrt{3x^2+2x+1}}$$

could be tackled with the substitutions we used in the previous example.

Multiple angle trigonometric identities can also be very useful: we have already made use of the formula

$$\cos 2\theta = 2\cos^2\theta - 1$$

to determine the integral of $\cos^2\theta$. Likewise, in principle ,we could integrate $\cos^n\theta$ by first writing it in terms of $\cos k\theta$ (for various $k$) — we will see how to do this in the class on complex numbers, but approach this integral in other ways in the next section on reduction formulae.

We close this section with a look at the $t$-substitution, which makes use of the *half-angle tangent formulas.* Faced with the integral

$$\int_0^\pi \frac{\mathrm{d}\theta}{2+\cos\theta},$$

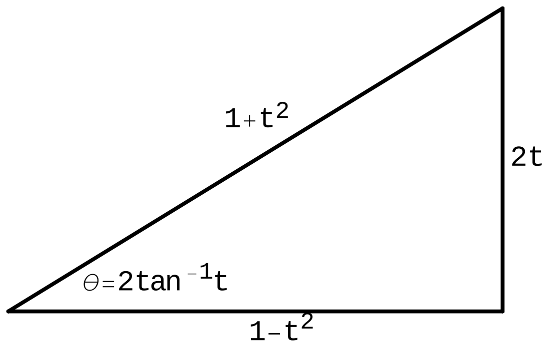
we make a substitution of the form

$$t = \tan\frac{\theta}{2}.$$

Each of the trigonometric functions sin, cos, tan can be written in terms of $t$. The formulae are

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}, \quad \tan\theta = \frac{2t}{1-t^2}.$$

An easy way to remember these formulae is probably by means of the right-angled triangle:




If we make this substitution in the above integral then firstly we need to note that $t = \tan(\theta/2)$ is a bijection from the range $[0, \pi)$ to the range $[0, \infty)$. Also

$$\mathrm{d}\theta = \mathrm{d}\left(2\tan^{-1} t\right) = \frac{2\ \mathrm{d}t}{1+t^2}.$$

So

$$\begin{aligned}
\int_0^\pi \frac{\mathrm{d}\theta}{2+\cos\theta} &= \int_0^\infty \frac{1}{2+\frac{1-t^2}{1+t^2}} \frac{2\ \mathrm{d}t}{1+t^2} \\
&= \int_0^\infty \frac{2\ \mathrm{d}t}{2+2t^2+1-t^2} \\
&= \int_0^\infty \frac{2\ \mathrm{d}t}{3+t^2} \\
&= \left[\frac{2}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}}\right]_0^\infty \\
&= \frac{2}{\sqrt{3}} \left(\frac{\pi}{2} - 0\right) \\
&= \frac{\pi}{\sqrt{3}}.
\end{aligned}$$

## 4 Reduction Formulae

In the previous section on integrating rational functions we were left with the problem of calculating

$$I_n = \int \cos^n \theta \; \mathrm{d}\theta,$$

and we will approach such integrals using *reduction formulae*. The idea is to write $I_n$ in terms of other $I_k$ where $k < n$, eventually reducing the problem to calculating $I_0$, or $I_1$ say, which are simple integrals.

Using IBP we see

$$\begin{aligned} I_n &= \int \cos^{n-1}\theta \times \cos\theta \; \mathrm{d}\theta \\ &= \cos^{n-1}\theta \sin\theta - \int (n-1)\cos^{n-2}\theta\,(-\sin\theta)\sin\theta \; \mathrm{d}\theta \\ &= \cos^{n-1}\theta\sin\theta + (n-1)\int \cos^{n-2}\theta\left(1-\cos^2\theta\right)\; \mathrm{d}\theta \\ &= \cos^{n-1}\theta\sin\theta + (n-1)\left(I_{n-2} - I_n\right). \end{aligned}$$

Rearranging this we see

$$I_n = \frac{\cos^{n-1}\theta\sin\theta}{n} + \frac{n-1}{n} I_{n-2}.$$

With this reduction formula $I_n$ can be rewritten in terms of simpler and simpler integrals until we are left only needing to calculate $I_0$, if $n$ is even, or $I_1$, if $n$ is odd — both these integrals are easy to calculate.

**Example 13** *Calculate*

$$I_7 = \int \cos^7\theta \; d\theta.$$

Using the reduction formula above

$$\begin{aligned} I_7 &= \frac{\cos^6\theta\sin\theta}{7} + \frac{6}{7} I_5 \\ &= \frac{\cos^6\theta\sin\theta}{7} + \frac{6}{7}\left(\frac{\cos^4\theta\sin\theta}{5} + \frac{4}{5} I_3\right) \\ &= \frac{\cos^6\theta\sin\theta}{7} + \frac{6\cos^4\theta\sin\theta}{35} + \frac{24}{35}\left(\frac{\cos^2\theta\sin\theta}{3} + \frac{2}{3} I_1\right) \\ &= \frac{\cos^6\theta\sin\theta}{7} + \frac{6\cos^4\theta\sin\theta}{35} + \frac{24\cos^2\theta\sin\theta}{105} + \frac{48}{105}\sin\theta + \text{const.} \end{aligned}$$

**Example 14** *Calculate*

$$\int_0^1 x^3 e^{2x} \; dx$$

This is an integral we previously calculated in the first section. We can approach this in a simpler, yet more general, fashion by setting up a reduction formula. For a natural number $n$ let

$$J_n = \int_0^1 x^n e^{2x} \; \mathrm{d}x$$

We can use then integration by parts to show

$$\begin{aligned} J_n &= \left[x^n \frac{e^{2x}}{2}\right]_0^1 - \int_0^1 n x^{n-1} \frac{e^{2x}}{2} \; \mathrm{d}x \\ &= \frac{e^2}{2} - \frac{n}{2} J_{n-1} \quad \text{if } n \geq 1. \end{aligned}$$

and so the calculation in Example 4 simplifies enormously (at least on the eye). We first note

$$J_0 = \int_0^1 e^{2x}\ \mathrm{d}x = \left[\frac{e^{2x}}{2}\right]_0^1 = \frac{e^2-1}{2},$$

and then applying the reduction formula:

$$\begin{aligned} J_3 &= \frac{e^2}{2} - \frac{3}{2}J_2 \\ &= \frac{e^2}{2} - \frac{3}{2}\left(\frac{e^2}{2} - \frac{2}{2}J_1\right) \\ &= \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3}{2}\left(\frac{e^2}{2} - \frac{1}{2}J_0\right) \\ &= \frac{e^2}{8} + \frac{3}{8}. \end{aligned}$$

Some integrands may involve two variables, such as:

**Example 15** *Calculate for positive integers $m, n$ the integral*

$$B\left(m,n\right) = \int_0^1 x^{m-1}\left(1-x\right)^{n-1}\ dx.$$

Calculating either $B\left(m,1\right)$ or $B\left(1,n\right)$ is easy; for example

$$B\left(m,1\right) = \int_0^1 x^{m-1}\ \mathrm{d}x = \frac{1}{m}. \tag{5}$$

So it would seem best to find a reduction formula that moves us towards either of these integrals. Using integration by parts, if $n \geq 2$ we have

$$\begin{aligned} B\left(m,n\right) &= \left[\frac{x^m}{m}\left(1-x\right)^{n-1}\right]_0^1 - \int_0^1 \frac{x^m}{m} \times (n-1) \times (-1)\left(1-x\right)^{n-2}\ \mathrm{d}x \\ &= 0 + \frac{n-1}{m}\int_0^1 x^m\left(1-x\right)^{n-2}\ \mathrm{d}x \\ &= \frac{n-1}{m}B\left(m+1,n-1\right). \end{aligned}$$

So if $n \geq 2$ we can apply this to see

$$\begin{aligned} B\left(m,n\right) &= \frac{n-1}{m}B\left(m+1,n-1\right) \\ &= \frac{n-1}{m} \times \frac{n-2}{m+1}B\left(m+2,n-2\right) \\ &= \left(\frac{n-1}{m}\right)\left(\frac{n-2}{m+1}\right)\cdots\left(\frac{1}{m+n-2}\right)B\left(m+n-1,1\right) \\ &= \left(\frac{n-1}{m}\right)\left(\frac{n-2}{m+1}\right)\cdots\left(\frac{1}{m+n-2}\right)\frac{1}{m+n-1} \\ &= \frac{(n-1)!}{(m+n-1)!/\left(m-1\right)!} \\ &= \frac{(m-1)!\,(n-1)!}{(m+n-1)!}. \end{aligned}$$

We have, in fact, already checked that this formula holds for the case $n = 1$ in equation (5).

# 5 Numerical Methods

Of course it's not always possible to calculate integrals exactly and there are numerical rules that will provide approximate values for integrals — approximate values, which by 'sampling' the function more and more times, can be made better and better.

Suppose that $f : [a, b] \to \mathbb{R}$ is the function we are wishing to integrate. Our idea will be to sample the function at $n + 1$ evenly spread points through the interval:

$$x_k = a + k\left(\frac{b-a}{n}\right) \quad \text{for } k = 0, 1, 2, \ldots, n,$$

so that $x_0 = a$ and $x_n = b$. The corresponding $y$-value we will denote as

$$y_k = f(x_k).$$

For ease of notation the width between each sample we will denote as

$$h = \frac{b-a}{n}.$$

There are various rules for making an estimate for the integrals of the function based on this data. We will consider the *Trapezium Rule* and *Simpson's Rule.*

- *Trapezium Rule.* This estimates the area as:

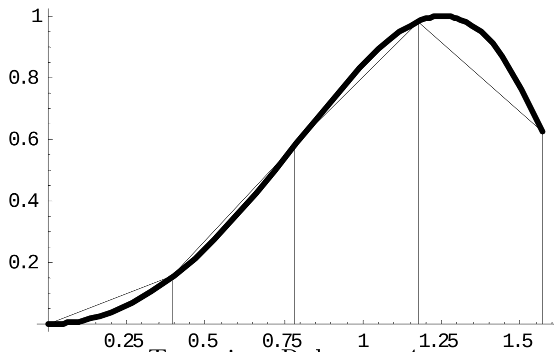
$$h\left(\frac{y_0}{2} + y_1 + y_2 + \cdots + y_{n-1} + \frac{y_n}{2}\right).$$

  This estimate is arrived at (as you might guess from the name) by approximating the area under the graph with trapezia. We presume that the graph behaves linearly between $(x_k, y_k)$ and $(x_{k+1}, y_{k+1})$ and take the area under the line segment connecting these points as our contribution.

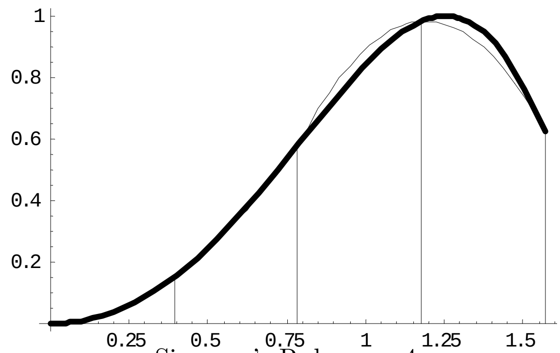
- *Simpson's Rule.* This requires that $n$ be even and estimates the area as:

$$\frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right).$$

  The more sophisticated Simpson's Rule works on the presumption that between the three points $(x_k, y_k), (x_{k+1}, y_{k+1}), (x_{k+2}, y_{k+2})$ (where $k$ is even) the function $f$ will change quadratically and it calculates the area contributed beneath each of these quadratic curves.




Trapezium Rule: $n = 4$




Simpson's Rule: $n = 4$

The above two graphs show applications of the trapezium rule and Simpson's rule in calculating

$$\int_0^{\pi/2} \sin\left(x^2\right)\,\mathrm{d}x$$

with $n = 4$ subintervals.

**Example 16** *Estimate the integral*

$$\int_0^1 x^3 \ dx$$

*using both the trapezium rule and Simpson's rule using* $2n$ *intervals.*

This is, of course, an integral we can calculate exactly as $1/4$. The two rules above give us:

$$\begin{aligned}
\text{Trapezium Approximation} &= \frac{1}{2n}\left(\frac{0^3}{2}+\left(\frac{1}{2n}\right)^3+\cdots+\left(\frac{2n-1}{2n}\right)^3+\frac{1^3}{2}\right) \\
&= \frac{1}{2n}\left(\frac{1}{8n^3}\sum_{k=1}^{2n-1}k^3+\frac{1}{2}\right) \\
&= \frac{1}{2n}\left(\frac{1}{8n^3}\times\frac{1}{4}(2n-1)^2(2n)^2+\frac{1}{2}\right) \\
&= \frac{4n^2+1}{16n^2} \\
&= \frac{1}{4}+\frac{1}{16n^2}.
\end{aligned}$$

and we also have

$$\begin{aligned}
\text{Simpson's Approximation} &= \frac{1}{6n}\left(0^3+4\left(\frac{1}{2n}\right)^3+2\left(\frac{2}{2n}\right)^3+\cdots+1^3\right) \\
&= \frac{1}{6n}\left(0+\frac{4}{(2n)^3}\sum_{k=1}^{2n-1}k^3-\frac{2}{(2n)^3}\sum_{k=1}^{n-1}(2k)^3+1\right) \\
&= \frac{1}{6n}\left(\frac{4}{8n^3}\times\frac{1}{4}(2n-1)^2(2n)^2-\frac{2}{8n^3}\times 8\times\frac{1}{4}(n-1)^2n^2+1\right) \\
&= \frac{3n^2}{12n^2} \\
&= \frac{1}{4}.
\end{aligned}$$

**Remark 17** *Note in these calculations we make use of the formula*

$$\sum_{k=1}^{n}k^3=\frac{1}{4}n^2(n+1)^2.$$

We see then that the error from the Trapezium Rule is $1/\left(16n^2\right)$ and so decreases very quickly. Amazingly Simpson's Rule does even better here and gets the answer spot on — the overestimates and underestimates of area from under these quadratics actually cancel out. In general Simpson's Rule is an improvement on the Trapezium Rule with the two errors (associated with $2n$ intervals) being given by:

$$\left|E_{\text{Trapezium}}\right| \leq \frac{(b-a)^3}{48n^2}\max\left\{\left|f''(x)\right| : a\leq x\leq b\right\},$$

and Simpson's Rule with $2n$ steps

$$\left|E_{\text{Simpson}}\right| \leq \frac{(b-a)^5}{2880n^4}\max\left\{\left|f^{(4)}(x)\right| : a\leq x\leq b\right\}.$$

Note that the error is $O\left(n^{-4}\right)$ for the Simpson Rule but only $O\left(n^{-2}\right)$ for the Trapezium Rule.