A SUBEXPONENTIAL QUANTUM ALGORITHM FOR THE SEMIDIRECT DISCRETE LOGARITHM PROBLEM

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Historical Disclaimer

Comparison with Recent Work

Not this work[1,2]	This work
Reduction to quantum-easy problems	Reduction to quantum-hard-ish problem
Works for some finite groups but not for semigroups	Works for any finite semigroup

[1] Imran and Ivanyos 2023.
[2] Mendelsohn, Dable-Heath, and Ling 2023.

Timeline

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2014-2021: design/analysis of different versions of an SDLP-based cryptosystem[3]

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Summer 2022: this work, first dedicated analysis of SDLP

↓

Spring 2023: applications of techniques in this paper to DSS[4]

↓

Christmas 2023: faster SDLP methods in some finite groups[5]

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[3] Habeeb, Kahrobaei, Koupparis, and Shpilrain 2014.

[4] B., Kahrobaei, Perret, and Shahandashti 2023.

[5] Imran and Ivanyos 2023; Mendelsohn, Dable-Heath, and Ling 2023.

SDLP

Definitions

Semidirect Product

Let G be a finite semigroup and $End(G)$ its semigroup of endomorphisms. We define $G \rtimes End(G)$ to be the semigroup of pairs in $G \times End(G)$ equipped with the following multiplication:

$$(g, \phi)(h, \psi) := (g\phi(h), \phi \circ \psi)$$

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Notice

$$(g, \phi)^2 = (g\phi(g), \phi^2)$$
$$(g, \phi)^3 = (g, \phi)(g\phi(g), \phi^2) = (g\phi(g)\phi^2(g), \phi^3)$$
$$(g, \phi)^4 = (g, \phi)(g\phi(g)\phi^2(g), \phi^3) = (g\phi(g)\phi^2(g)\phi^3(g), \phi^4)$$

Definitions

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Semidirect Exponentiation

Fix $(g, \phi) \in G \rtimes End(G)$. Define $s_{g,\phi} : \mathbb{N} \to G$ to be the group element such that

$$(g, \phi)^x = (s_{g,\phi}(x), \phi^x)$$

We have seen that

$$s_{g,\phi}(x) = g\phi(g)...\phi^{x-1}(g)$$

SDLP

Fix $G \rtimes End(G)$ and a pair (g, ϕ). Suppose we are given $s_{g,\phi}(x)$ for some $x \in \mathbb{N}$. The **S**emidirect **D**iscrete **L**ogarithm **P**roblem is to recover x.

Examples

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Let $G = M_3(\mathbb{Z}_3)$, $A = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$, $\phi_B(M) = BMB^{-1}$.

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$$\ldots$$

$$s_{A,\phi_B}(10) = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = s_{A,\phi_B}(2)$$

A Group Action

The $*$ Operator

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$$
\begin{aligned}
(s_{g,\phi}(x+y), \phi^{x+y}) = (g,\phi)^{x+y} &= (g,\phi)^x (g,\phi)^y \\
&= (s_{g,\phi}(x), \phi^x)(s_{g,\phi}(y), \phi^y) \\
&= (s_{g,\phi}(x)\phi^x(s_{g,\phi}(y)), \phi^{x+y})
\end{aligned}
$$

so $s_{g,\phi}(x + y) = s_{g,\phi}(x)\phi^x(s_{g,\phi}(y))$. We can add in the argument of $s_{g,\phi}$.

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$*$

Let $\mathcal{X}_{g,\phi} = \{s_{g,\phi}(i) : i \in \mathbb{N}\}$, and define $* : \mathbb{N} \times \mathcal{X}_{g,\phi} \to \mathcal{X}_{g,\phi}$ by

$$x * s_{g,\phi}(y) = s_{g,\phi}(x)\phi^x(s_{g,\phi}(y))$$

We have $x * s_{g,\phi}(y) = s_{g,\phi}(x+y)$.

Shape of $\mathcal{X}_{g,\phi}$

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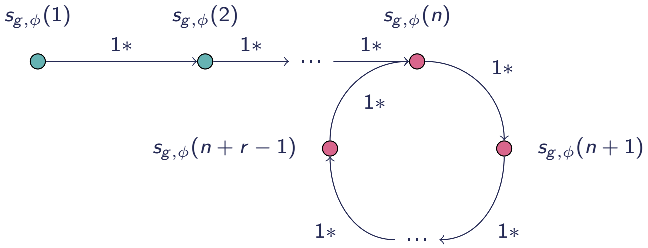
Set $\mathcal{X}_{g,\phi} = \{s_{g,\phi}(i) : i \in \mathbb{N}\}$.



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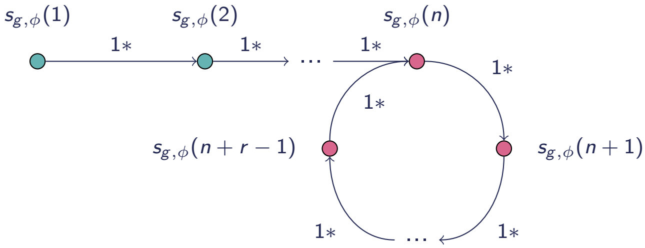

Terminology

We call n the **index**, r the **period**, $\{g, ..., s_{g,\phi}(n-1)\}$ the **tail**, and $\{s_{g,\phi}(n), ..., s_{g,\phi}(n+r-1)\}$ the **cycle**.

Definitions

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Finite Group Action

Let G be a finite group, X be a finite set and $*$ be a function $* : G \times X \to X$. The tuple $(G, X, *)$ is a **group action** if

$$1_G * x = x \text{ for each } x \in X$$

$$(gh) * x = g * (h * x) \text{ for each } g, h \in G,\ x \in X$$

Vectorisation[6]/Group Action DLog

Let $(G, X, *)$ be a group action. Given $x, y \in X$, the **vectorisation problem** is to find a g (if one exists) such that $g * x = y$.

[6] Couveignes 2006.

Contributions

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Theorem [B., Kahrobaei, Perret, Shahandashti]

Let G be a finite semigroup and consider the semigroup $G \rtimes End(G)$. Fix a pair $(g, \phi) \in G \rtimes End(G)$, and let $\mathcal{C}_{g,\phi}$ denote the corresponding cycle. The tuple $(\mathbb{Z}_r, \mathcal{C}_{g,\phi}, \circledast)$ is a free, transitive group action, where r, the period associated to (g, ϕ), is $|\mathcal{C}_{g,\phi}|$.

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Theorem [B., Kahrobaei, Perret, Shahandashti]

There is a fast quantum reduction from SDLP w.r.t (g, ϕ) to a vectorisation problem, and therefore quantum algorithms for SDLP of quantum complexity $2^{\mathcal{O}(\sqrt{\log r})}$, where r is the period associated to (g, ϕ).

The Reduction

Background

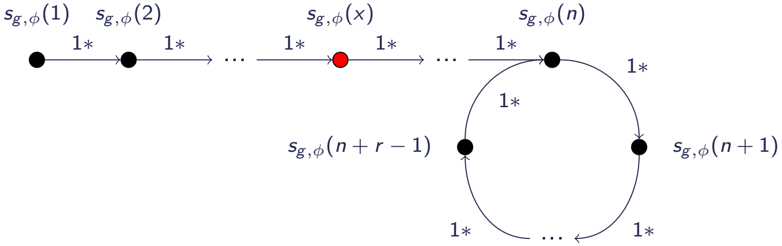
Well-known that the Vectorisation Problem reduces to dihedral hidden subgroup problem.[7]

Dihedral hidden subgroup problem admits (a) quantum algorithm with complexity $2^{\mathcal{O}(\sqrt{\log n})}$ for D_{2n}.[8]

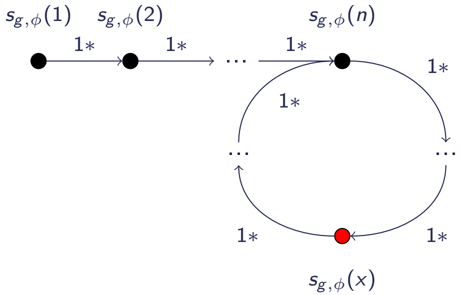
Reduction of Semigroup DLog to a DLog problem has to address a similar structure to us.[9]

[7] Childs, Jao, and Soukharev 2014.
[8] Kuperberg 2005.
[9] Childs and Ivanyos 2014.

Scenario 1: $x < n$



Scenario 2: $x \geq n$



Roadmap Given n, r

Suppose we are given n, r.

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Suppose we are given n, r.
Notice that $r * s_{g,\phi}(x) = s_{g,\phi}(x) \iff s_{g,\phi}(x) \in \mathcal{C}_{g,\phi}$

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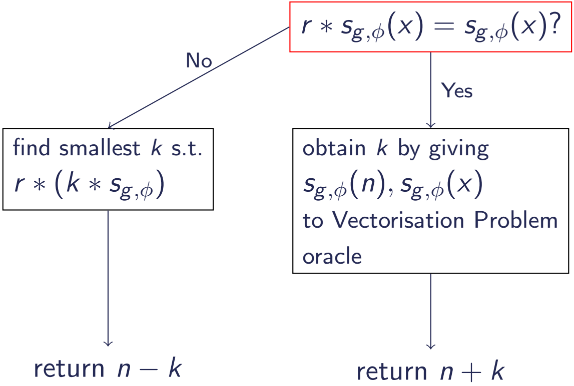

Computing n, r

Given r compute n as the smallest integer such that $r * s_{g,\phi}(n) = s_{g,\phi}(n)$.

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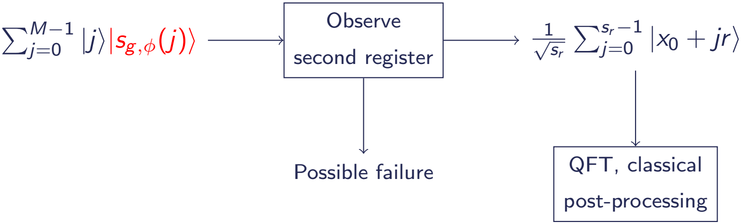

Conclusions

Takeaways and Open Problems

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One can solve SDLP for (g, ϕ) in quantum time $2^{\mathcal{O}(\sqrt{\log r})}$ where r is a function of g, ϕ - not much known about its size.

In the generic case this remains state-of-the-art; possible that specific semigroups would yield faster results

Fast classical methods of computing n, r might give us interesting crypto.

Further Reading

Fast SDLP now resolved for *all** finite groups.

https://eprint.iacr.org/2024/905

More on group-based cryptography:

http://aimpl.org/postquantgroup/ .

*up to constructive recognition.