MATHEMATICS ADMISSIONS TEST

For candidates applying for Mathematics, Computer Science or one of their joint degrees at OXFORD UNIVERSITY and/or IMPERIAL COLLEGE LONDON and/or UNIVERSITY OF WARWICK

Thursday 2 November 2017

Time Allowed: 2½ hours

Please complete the following details in BLOCK CAPITALS. You must use a pen.

Surname	
Other names	

Candidate Number	**M**					

This paper contains 7 questions of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

A: Oxford Applicants: if you are applying to Oxford for the degree course:

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics, you should attempt Questions **1,2,3,4,5.**
- Mathematics & Computer Science, you should attempt Questions **1,2,3,5,6.**
- Computer Science *or* Computer Science & Philosophy, you should attempt **1,2,5,6,7.**

Directions under A take priority over any directions in B which are relevant to you.

B: Imperial or Warwick Applicants: if you are applying to the University of Warwick for Mathematics BSc, Master of Mathematics, or if you are applying to Imperial College for any of the Mathematics courses**:** Mathematics, Mathematics (Pure Mathematics), Mathematics with a Year in Europe, Mathematics with Applied Mathematics/Mathematical Physics, Mathematics with Mathematical Computation, Mathematics with Statistics, Mathematics with Statistics for Finance, Mathematics Optimisation and Statistics, you should attempt Questions **1,2,3,4,5.**

Further credit cannot be obtained by attempting extra questions. **Calculators are not permitted.**

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

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Please complete these details below in block capitals.

Centre Number															
Candidate Number	M														
UCAS Number (if known)				–				–							

	d	d		m	m		y	y
Date of Birth			–			–		

Please tick the appropriate box:

- [ ] I have attempted Questions **1,2,3,4,5**
- [ ] I have attempted Questions **1,2,3,5,6**
- [ ] I have attempted Questions **1,2,5,6,7**

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Q1	Q2	Q3	Q4	Q5	Q6	Q7

1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given **five** possible answers, just one of which is correct. Indicate for each part **A-J** which answer (a), (b), (c), (d), or (e) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)	(e)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

A. Let

$$f(x) = 2x^3 - kx^2 + 2x - k.$$

For what values of the real number k does the graph $y = f(x)$ have two distinct real stationary points?

(a) $-2\sqrt{3} < k < 2\sqrt{3}$
(b) $k < -2\sqrt{3}$ or $2\sqrt{3} < k$
(c) $k < -\sqrt{21} - 3$ or $\sqrt{21} - 3 < k$
(d) $-\sqrt{21} - 3 < k < \sqrt{21} - 3$
(e) all values of k.

B. The minimum value achieved by the function

$$f(x) = 9\cos^4 x - 12\cos^2 x + 7$$

equals

(a) 3 (b) 4 (c) 5 (d) 6 (e) 7.

Turn over

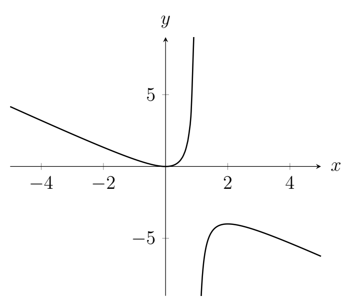
C. A sequence (a_n) has the property that

$$a_{n+1} = \frac{a_n}{a_{n-1}}$$

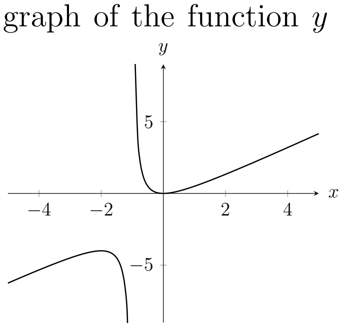
for every $n \geqslant 2$. Given that $a_1 = 2$ and $a_2 = 6$, what is a_{2017}?

(a) $\dfrac{1}{6}$ (b) $\dfrac{1}{3}$ (c) $\dfrac{1}{2}$ (d) 2 (e) 3.

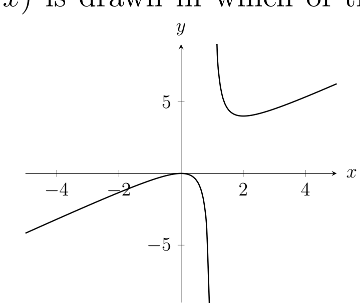
D. The diagram below shows the graph of $y = f(x)$.



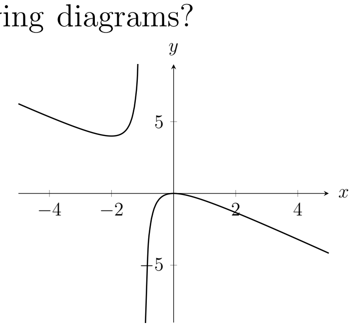
The graph of the function $y = -f(-x)$ is drawn in which of the following diagrams?



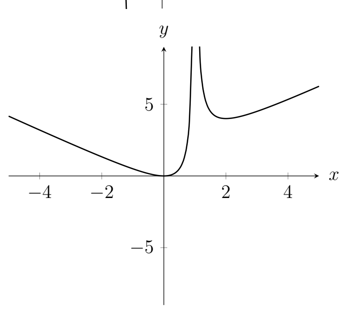
(a)



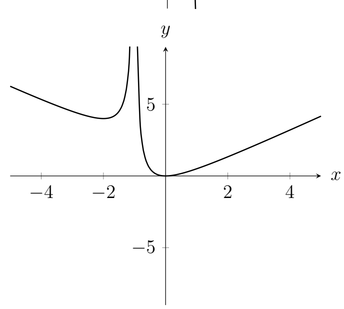
(b)



(c)



(d)



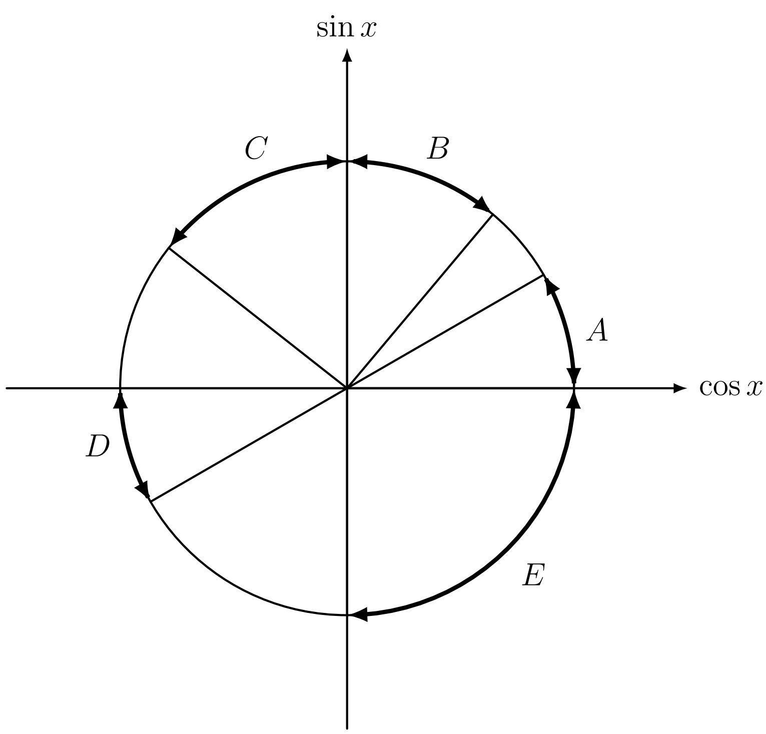
(e)

E. Let a and b be *positive integers* such that $a + b = 20$. What is the maximum value that $a^2 b$ can take?

(a) 1000 (b) 1152 (c) 1176 (d) 1183 (e) 1196.

F. The picture below shows the unit circle, where each point has coordinates $(\cos x, \sin x)$ for some x. Which of the marked arcs corresponds to

$$\tan x < \cos x < \sin x \ ?$$



(a) A (b) B (c) C (d) D (e) E.

Turn over

G. For all θ in the range $0 \leqslant \theta < 2\pi$ the line

$$(y-1)\cos\theta = (x+1)\sin\theta$$

divides the disc $x^2 + y^2 \leqslant 4$ into two regions. Let $A(\theta)$ denote the area of the larger region.

Then $A(\theta)$ achieves its maximum value at

(a) one value of θ (b) two values of θ (c) three values of θ

(d) four values of θ (e) all values of θ.

H. In this question a and b are real numbers, and a is non-zero.
When the polynomial $x^2 - 2ax + a^4$ is divided by $x + b$ the remainder is 1.
The polynomial $bx^2 + x + 1$ has $ax - 1$ as a factor.

It follows that b equals

(a) 1 only (b) 0 or -2 (c) 1 or 2 (d) 1 or 3 (e) -1 or 2.

I. Let $a, b, c > 0$ and $a \neq 1$. The equation

$$\log_b\left((b^x)^x\right) + \log_a\left(\frac{c^x}{b^x}\right) + \log_a\left(\frac{1}{b}\right)\log_a(c) = 0$$

has a repeated root when

(a) $b^2 = 4ac$ (b) $b = \dfrac{1}{a}$ (c) $c = \dfrac{b}{a}$ (d) $c = \dfrac{1}{b}$ (e) $a = b = c$.

J. Which of these integrals has the largest value? You are not expected to calculate the exact value of any of these.

(a) $\displaystyle\int_0^2 (x^2 - 4)\sin^8(\pi x)\,\mathrm{d}x$ (b) $\displaystyle\int_0^{2\pi} (2 + \cos x)^3\,\mathrm{d}x$ (c) $\displaystyle\int_0^{\pi} \sin^{100} x\,\mathrm{d}x$

(d) $\displaystyle\int_0^{\pi} (3 - \sin x)^6\,\mathrm{d}x$ (e) $\displaystyle\int_0^{8\pi} 108(\sin^3 x - 1)\,\mathrm{d}x$.

Turn over

2. For **ALL APPLICANTS.**

There is a unique real number α that satisfies the equation

$$\alpha^3 + \alpha^2 = 1.$$

[You are not asked to prove this.]

(i) Show that $0 < \alpha < 1$.

(ii) Show that

$$\alpha^4 = -1 + \alpha + \alpha^2.$$

(iii) Four functions of α are given in (a) to (d) below. In a similar manner to part (ii), each is equal to a quadratic expression

$$A + B\alpha + C\alpha^2$$

in α, where A, B, C are integers. (So in (ii) we found $A = -1$, $B = 1$, $C = 1$.) You may assume in each case that the quadratic expression is unique.

In each case below find the quadratic expression in α.

(a) α^{-1}.

(b) The infinite sum

$$1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4 - \alpha^5 + \cdots .$$

(c) $(1 - \alpha)^{-1}$.

(d) The infinite product

$$(1 + \alpha)(1 + \alpha^2)(1 + \alpha^4)(1 + \alpha^8)(1 + \alpha^{16}) \cdots .$$

Turn over
If you require additional space please use the pages at the end of the booklet

3.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Computer Science and *Computer Science & Philosophy* **applicants should turn to page 14.**

For each positive integer k, let $f_k(x) = x^{1/k}$ for $x \geqslant 0$.

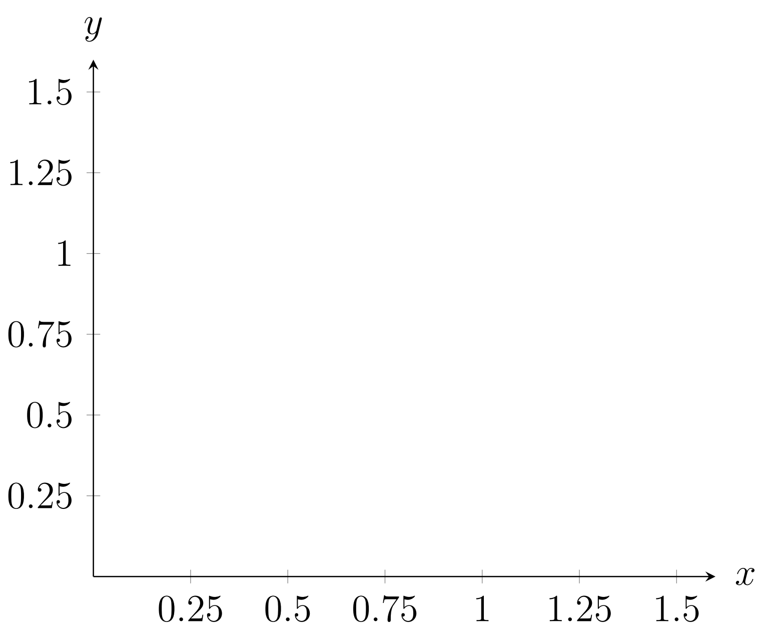
(i) On the same axes (provided below), labelling each curve clearly, sketch $y = f_k(x)$ for $k = 1, 2, 3$, indicating the intersection points.

(ii) Between the two points of intersection in (i), the curves $y = f_k(x)$ enclose several regions. What is the area of the region between $y = f_k(x)$ and $y = f_{k+1}(x)$? Verify that the area of the region between $y = f_1(x)$ and $y = f_2(x)$ is $\frac{1}{6}$.

Let c be a constant where $0 < c < 1$.

(iii) Find the x-coordinates of the points of intersection of the line $y = c$ with $y = f_1(x)$ and of $y = c$ with $y = f_2(x)$.

(iv) The constant c is chosen so that the line $y = c$ divides the region between $y = f_1(x)$ and $y = f_2(x)$ into two regions of equal area. Show that c satisfies the cubic equation $4c^3 - 6c^2 + 1 = 0$. Hence find c.



Turn over
If you require additional space please use the pages at the end of the booklet

4.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

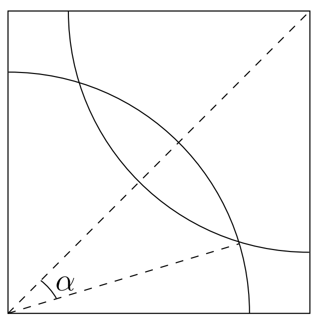
Mathematics & Computer Science, *Computer Science* and *Computer Science & Philosophy* **applicants should turn to page 14.**

A horse is attached by a rope to the corner of a square field of side length 1.

(i) What length of rope allows the horse to reach precisely half the area of the field?

Another horse is placed in the field, attached to the corner diagonally opposite from the first horse. Each horse has a length of rope such that each can reach half the field.

(ii) Explain why the area that both can reach is the same as the area neither can reach.



(iii) The angle α is marked in the diagram above. Show that $\alpha = \cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right)$ and hence show that the area neither can reach is $\frac{4}{\pi}\cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right) - \sqrt{\frac{4-\pi}{\pi}}$. Note that $\cos^{-1}$ can also be written as arccos.

A third horse is placed in the field, and the three horses are rearranged. One horse is now attached to the midpoint of the bottom side of the field, and another horse is now attached to the midpoint of the left side of the field. The third horse is attached to the upper right corner.

(iv) Given each horse can access an equal area of the field and that none of the areas overlap, what length of rope must each horse have to minimise the area that no horse can reach?

The horses on the bottom and left midpoints of the field are each replaced by a goat; each goat is attached by a rope of length g to the same midpoint as in part (iii). The remaining horse is attached to the upper right corner with rope length h.

(v) Given that $0 \leqslant h \leqslant 1$, and that none of the animals' areas can overlap, show that $\frac{\sqrt{5}-2}{2} \leqslant g \leqslant \frac{1}{2\sqrt{2}}$ holds if the area that the animals can reach is maximised.

Turn over

If you require additional space please use the pages at the end of the booklet

5. For **ALL APPLICANTS.**

Ten children, $c_0, c_1, c_2, \ldots, c_9$, are seated clockwise in a circle. The teacher walks clockwise behind the children with a large bag of sweets. She gives a sweet to child c_1. She then skips a child and gives a sweet to the next child, c_3. Next she skips two children and gives a sweet to the next child, c_6. She continues in this way, at each stage skipping one more child than at the preceding stage before giving a sweet to the next child.

(i) The kth sweet is given to child c_i. Explain why i is the last digit of the number

$$\frac{k(k+1)}{2}.$$

(ii) Let $1 \leqslant k \leqslant 18$. Explain why the kth and $(20-k-1)$th sweets are given to the same child.

(iii) Explain why the kth sweet is given to the same child as the $(k+20)$th sweet.

(iv) Which children can never receive any sweets?

When the teacher has given out all the sweets, she has walked exactly 183 times round the circle, and given the last sweet to c_0.

(v) How many sweets were there initially?

(vi) Which children received the most sweets and how many did they receive?

Turn over

If you require additional space please use the pages at the end of the booklet

6.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{COMPUTER SCIENCE}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

You need to pack several items into your shopping bag, without squashing any item. Suppose each item i has a (positive) weight w_i, and a strength s_i which is the maximum weight that can be placed above it without it being squashed. For the purposes of this question, suppose that the items will be arranged one on top of the other within your bag. We will say that a particular packing order is *safe* if no item is squashed, that is, for each item i, s_i is at least the sum of the w_j corresponding to items j placed above item i. For example, suppose we have the following items, packed in the order given

Ordering	Item	w_i	s_i
Top	Apples	5	6
Middle	Bread	4	4
Bottom	Carrots	12	9

This packing is not safe: the bread is squashed, since the weight above it (5) is greater than its strength (4). However, swapping the apples and the bread gives a safe packing.

(i) Which of the other four orderings of apples, bread, and carrots are safe or unsafe?

(ii) Consider the tactic of packing the items in weight order, with the heaviest at the bottom. Show by giving an example that this might not produce a safe packing order, even if a safe packing order exists.

(iii) Now consider the tactic of packing the items in strength order, with the strongest at the bottom. Again show by giving an example that this might not produce a safe packing order, even if one exists.

(iv) Suppose we have a safe packing order, with item j *directly* on top of item i. Suppose further that

$$w_j - s_i \;\geqslant\; w_i - s_j.$$

Show that if we swap items i and j, we still have a safe packing order.

(v) Hence suggest a practical method of producing a safe packing order if one exists. Explain why your method works. (Listing all possible orderings is not practical.)

Turn over
If you require additional space please use the pages at the end of the booklet

7.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{COMPUTER SCIENCE} \\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY} \end{array} \right\}$ **ONLY.**

A simple computer can operate on lists of numbers in several ways.

- Given two lists a and b, it can make the *join* $a + b$, by placing list b after list a. For example if

$$a = (1,2,3,4) \quad \text{and} \quad b = (5,6,7) \quad \text{then} \quad a + b = (1,2,3,4,5,6,7).$$

- Given a list a it can form the *reverse* sequence $R(a)$ by listing a in reverse order. For example if

$$a = (1,2,3,4) \quad \text{then} \quad R(a) = (4,3,2,1).$$

(i) Given sequences a and b, express $R(a + b)$ as the join of two sequences. What is $R(R(a))$?

- Given a sequence a of length n and $0 \leqslant k \leqslant n$, then the kth *shuffle* S_k of a moves the first k elements of a to the end of the sequence in reverse order. For example

$$S_2(1,2,3,4,5) = (3,4,5,2,1) \quad \text{and} \quad S_3(1,2,3,4,5) = (4,5,3,2,1).$$

(ii) Given two sequences a and b, both of length k, express $S_k(a + b)$ as the join of two sequences. What is $S_k(S_k(a + b))$?

(iii) Now let $a = (1,2,3,4,5,6,7,8)$. Write down

$$S_5(S_5(a))$$

as the join of three sequences that are either in order or in reverse order. Show that the sequence a is back in its original order after four S_5 shuffles.

(iv) Now let a be a sequence of length n with $k \geqslant n/2$. Prove, after S_k is performed four times, that the sequence returns to its original order.

(v) Give an example to show that when $k < n/2$, the sequence need not be in its original order after S_k is performed four times. For your example how many times must S_k be performed to first return the sequence to its original order?

End of last question

Turn over
If you require additional space please use the pages at the end of the booklet

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