MATHEMATICS ADMISSIONS TEST

For candidates applying for Mathematics, Computer Science or one of their joint degrees at OXFORD UNIVERSITY and/or IMPERIAL COLLEGE LONDON and/or UNIVERSITY OF WARWICK

Wednesday 31 October 2018

Time Allowed: 2½ hours

Please complete the following details in BLOCK CAPITALS. You must use a pen.

Surname	
Other names	

Candidate Number	**M**					

This paper contains 7 questions of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

A: Oxford Applicants: if you are applying to Oxford for the degree course:

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics, you should attempt Questions **1,2,3,4,5.**
- Mathematics & Computer Science, you should attempt Questions **1,2,3,5,6.**
- Computer Science *or* Computer Science & Philosophy, you should attempt **1,2,5,6,7.**

Directions under A take priority over any directions in B which are relevant to you.

B: Imperial or Warwick Applicants: if you are applying to the University of Warwick for Mathematics BSc, Master of Mathematics, or if you are applying to Imperial College for any of the Mathematics courses**:** Mathematics, Mathematics (Pure Mathematics), Mathematics with a Year in Europe, Mathematics with Applied Mathematics/Mathematical Physics, Mathematics with Mathematical Computation, Mathematics with Statistics, Mathematics with Statistics for Finance, Mathematics Optimisation and Statistics, you should attempt Questions **1,2,3,4,5.**

Further credit cannot be obtained by attempting extra questions. **Calculators are not permitted.**

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

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Wednesday 31 October 2018

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Please complete these details below in block capitals.

Centre Number													
Candidate Number	**M**												
UCAS Number (if known)				–				–					

	d	d		m	m		y	y
Date of Birth			–			–		

Please tick the appropriate box:

- ☐ I have attempted Questions **1,2,3,4,5**
- ☐ I have attempted Questions **1,2,3,5,6**
- ☐ I have attempted Questions **1,2,5,6,7**

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Q1	Q2	Q3	Q4	Q5	Q6	Q7

1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given **five** possible answers, just one of which is correct. Indicate for each part **A-J** which answer (a), (b), (c), (d), or (e) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)	(e)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

A. The area of the region bounded by the curve $y = \sqrt{x}$, the line $y = x - 2$ and the x-axis equals

(a) 2, (b) $\frac{5}{2}$, (c) 3, (d) $\frac{10}{3}$, (e) $\frac{16}{3}$.

B. The function $y = e^{kx}$ satisfies the equation

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - y\right) = y\frac{\mathrm{d}y}{\mathrm{d}x}$$

for

(a) no values of k,
(b) exactly one value of k,
(c) exactly two distinct values of k,
(d) exactly three distinct values of k,
(e) infinitely many distinct values of k.

Turn over

C. Let a, b, c and d be real numbers. The two curves $y = ax^2 + c$ and $y = bx^2 + d$ have exactly two points of intersection precisely when

(a) $\frac{a}{b} < 1$, (b) $\frac{a}{b} < \frac{c}{d}$, (c) $a < b$, (d) $c < d$, (e) $(d-c)(a-b) > 0$.

D. If $f(x) = x^2 - 5x + 7$, what are the coordinates of the minimum of $y = f(x-2)$?

(a) $\left(\frac{5}{2}, \frac{3}{4}\right)$, (b) $\left(\frac{9}{2}, \frac{3}{4}\right)$, (c) $\left(\frac{1}{2}, \frac{3}{4}\right)$, (d) $\left(\frac{9}{2}, \frac{-5}{4}\right)$, (e) $\left(\frac{5}{2}, \frac{-5}{4}\right)$.

E. A circle of radius 2, centred on the origin, is drawn on a grid of points with integer coordinates. Let n be the number of grid points that lie within or on the circle. What is the smallest amount the radius needs to increase by for there to be $2n-5$ grid points within or on the circle?

(a) $\sqrt{5}-2$, (b) $\sqrt{6}-2$, (c) $\sqrt{8}-2$, (d) 1, (e) $\sqrt{8}$.

F. A particle moves in the xy-plane, starting at the origin $(0,0)$. At each turn, the particle may move in one of two ways:

- it may move two to the right and one up, that is, it may be translated by the vector $(2,1)$, or
- it may move one to the right and two up, that is, it may be translated by the vector $(1,2)$.

What is the closest the particle may come to the point $(25,75)$?

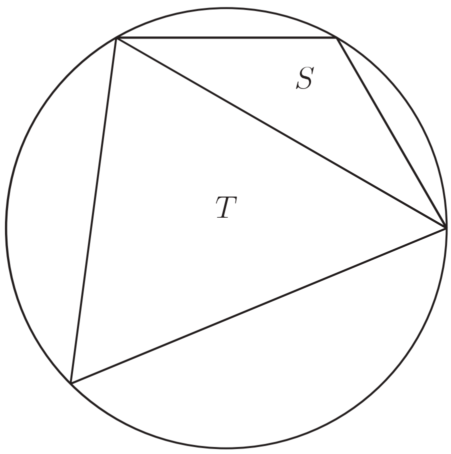
(a) 0, (b) $5\sqrt{5}$, (c) $2\sqrt{53}$, (d) 25, (e) 35.

Turn over

G. The parabolas with equations $y = x^2 + c$ and $y^2 = x$ touch (that is, meet tangentially) at a single point. It follows that c equals

(a) $\dfrac{1}{2\sqrt{3}}$, (b) $\dfrac{3}{4\sqrt[3]{4}}$, (c) $\dfrac{-1}{2}$, (d) $\sqrt{5} - \sqrt{3}$, (e) $\sqrt{\dfrac{2}{3}}$.

H. Two triangles S and T are inscribed in a circle, as shown in the diagram below.



The triangles have respective areas s and t and S is the smaller triangle so that $s < t$. The smallest value that

$$\frac{4s^2 + t^2}{5st}$$

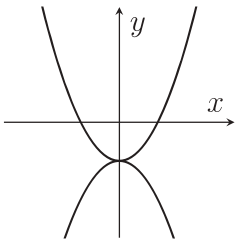
can equal is

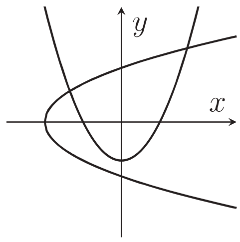
(a) $\dfrac{2}{5}$, (b) $\dfrac{3}{5}$, (c) $\dfrac{4}{5}$, (d) 1, (e) $\dfrac{3}{2}$.

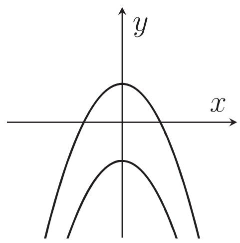
I. A sketch of the curve

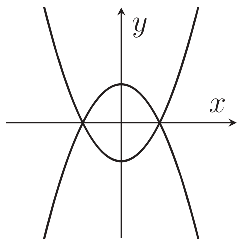
$$(x^8 + 4yx^6 + 6y^2x^4 + 4y^3x^2 + y^4)^2 = 1$$

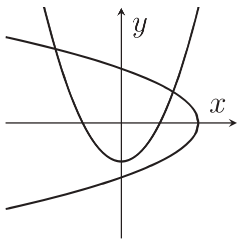
is given below in

(a) 

(b) 

(c) 

(d) 

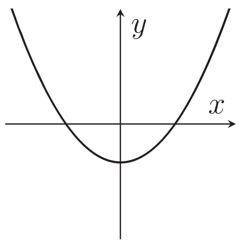
(e) 

J. Which of the following could be the sketch of a curve

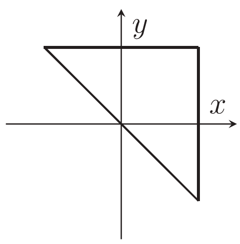
$$p(x) + p(y) = 0$$

for some polynomial p?

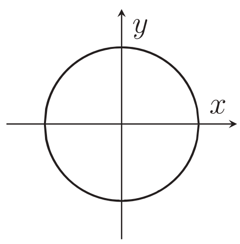
A



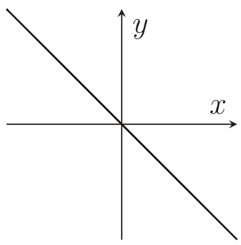
B



C



D



(a) A and D, but not B or C;
(b) A and B, but not C or D;
(c) C and D, but not A or B;
(d) A, C and D, but not B;
(e) A, B and C, but not D.

Turn over

2. For **ALL APPLICANTS.**

Let S and T denote transformations of the xy-plane

$$S(x, y) = (x + 1, y), \qquad T(x, y) = (-y, x).$$

We will write, for example, TS to denote the composition of applying S then T, that is

$$TS(x, y) = T(S(x, y)),$$

and write T^n to denote n applications of T where n is a positive integer.

(i) Show that $TS(x, y) \neq ST(x, y)$.

(ii) For what values of n is it the case that $T^n(x, y) = (x, y)$ for all x, y?

(iii) Show that applications of S and T in some order can produce the transformation

$$U(x, y) = (x - 1, y).$$

What is the least number of applications (of S and T in total) that can produce U? Justify your answer.

(iv) Show that for any integers a and b there is some sequence of applications of S and T that maps $(0, 0)$ to (a, b).

(v) The parabola C has equation $y = x^2 + 2x + 2$.
What is the equation of the curve obtained by applying S to C?
What is the equation of the curve obtained by applying T to C?

Turn over

If you require additional space please use the pages at the end of the booklet

3.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Computer Science and *Computer Science & Philosophy* **applicants should turn to page 14.**

Let $g(x)$ be the function defined by

$$g(x) = \begin{cases} (x-1)^2 + 1 & \text{if } x \geqslant 0 \\ 3 - (x+1)^2 & \text{if } x \leqslant 0, \end{cases}$$

and for $x \neq 0$ write $m(x)$ for the gradient of the chord between $(0, g(0))$ and $(x, g(x))$.

(i) Sketch the graph $y = g(x)$ for $-3 \leqslant x \leqslant 3$.

(ii) Write down expressions for $m(x)$ in the two cases $x \geqslant 0$ and $x < 0$.

(iii) Show that $m(x) + 2 = x$ for $x > 0$. What is the value of $m(x) + 2$ when $x < 0$?

(iv) Explain why g has derivative -2 at 0.

(v) Suppose that $p < q$ and that $h(x)$ is a cubic with a maximum at $x = p$ and a minimum at $x = q$. Show that $h'(x) < 0$ whenever $p < x < q$.

Suppose that c and d are real numbers and that there is a cubic $h(x)$ with a maximum at $x = -1$ and a minimum at $x = 1$ such that $h'(0) = -3c$ and $h(0) = d$.

(vi) Show that $c > 0$ and find a formula for $h(x)$ in terms of c and d (and x).

(vii) Show that there are no values of c and d such that the graphs of $y = g(x)$ and $y = h(x)$ are the same for $-3 \leqslant x \leqslant 3$.

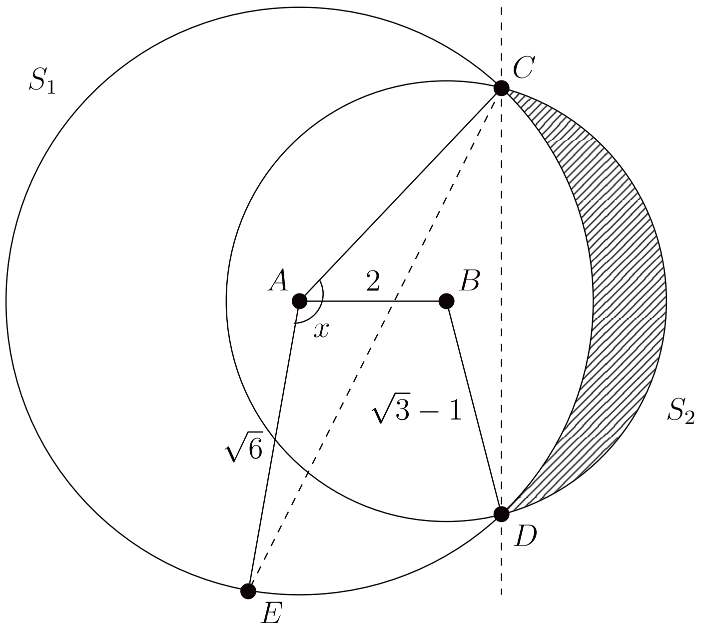
Turn over
If you require additional space please use the pages at the end of the booklet

4.

For **APPLICANTS IN** { **MATHEMATICS** / **MATHEMATICS & STATISTICS** / **MATHEMATICS & PHILOSOPHY** } **ONLY.**

Mathematics & Computer Science, *Computer Science* and *Computer Science & Philosophy* **applicants should turn to page 14.**

Consider two circles S_1 and S_2 centred at A and B and with radii $\sqrt{6}$ and $\sqrt{3}-1$, respectively. Suppose that the two circles intersect at two distinct points C and D. Suppose further that the two centres A and B are of distance 2 apart. The sketch below is not to scale.



(i) Find the angle $\angle CBA$, and deduce that A and B lie on the same side of the line CD.

(ii) Show that CD has length $3-\sqrt{3}$ and hence calculate the angle $\angle CAD$.

(iii) Show that the area of the region lying inside the circle S_2 and outside of the circle S_1 (that is the shaded region in the picture) is equal to

$$\frac{\pi}{6}(5-4\sqrt{3})+3-\sqrt{3}.$$

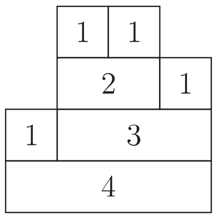
(iv) Suppose that a line through C is drawn such that the total area covered by S_1 and S_2 is split into two equal areas. Let E be the intersection of this line with S_1 and x denote the angle $\angle CAE$. You may assume that E lies on the larger arc CD of S_1. Write down an equation which x satisfies and explain why there is a unique solution x.

Turn over

If you require additional space please use the pages at the end of the booklet

5. For **ALL APPLICANTS.**

Let n be a positive integer. An *n-brick* is a rectangle of height 1 and width n. A *1-tower* is defined as a 1-brick. An *n-tower*, for $n \geqslant 2$, is defined as an n-brick on top of which exactly two other towers are stacked: a k_1-tower and a k_2-tower such that $1 \leqslant k_1 \leqslant n-1$ and $k_1 + k_2 = n$. The k_1-tower is placed to the left of the k_2-tower so that side-by-side they fit exactly on top of the n-brick. For example, here is a 4-tower:



(i) Draw the four other 4-towers.

(ii) What is the maximum height of an n-tower? Justify your answer.

(iii) The *area* of a tower is defined as the sum of the widths of its bricks. For example, the 4-tower drawn above has area $4+4+3+2 = 13$. Give an expression for the area of an n-tower of maximum height.

(iv) Show that there are infinitely many n such that there is an n-tower of height exactly $1 + \log_2 n$.

(v) Write t_n for the number of n-towers. We have $t_1 = 1$. For $n \geqslant 2$ give a formula for t_n in terms of t_k for $k < n$. Use your formula to compute t_6.

(vi) Show that t_n is odd if and only if t_{2n} is odd.

Turn over

If you require additional space please use the pages at the end of the booklet

6.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{COMPUTER SCIENCE}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

A positive rational number q is expressed in *friendly form* if it is written as a finite sum of reciprocals of distinct positive integers. For example, $\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20}$.

(i) Express the following numbers in friendly form: $\frac{2}{3}$, $\frac{2}{5}$, $\frac{23}{40}$.

(ii) Let q be a rational number with $0 < q < 1$, and m be the smallest natural number such than $\dfrac{1}{m} \leqslant q$. Suppose $q = \dfrac{a}{b}$ and $q - \dfrac{1}{m} = \dfrac{c}{d}$ in their lowest terms. Show that $c < a$.

(iii) Suggest a procedure by which any rational q with $0 < q < 1$ can be expressed in friendly form. Use the result in part (ii) to show that the procedure always works, generating distinct reciprocals and finishing within a finite time.

(iv) Demonstrate your procedure by finding a friendly form for $\frac{4}{13}$.

(v) Assuming that $\sum_{n=1}^{N} \dfrac{1}{n}$ increases without bound as N becomes large, show that every positive rational number can be expressed in friendly form.

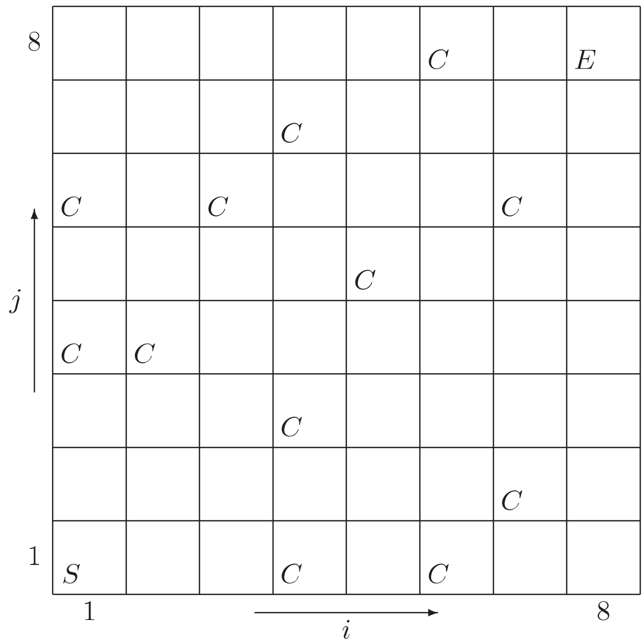
Turn over

If you require additional space please use the pages at the end of the booklet

7.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{COMPUTER SCIENCE} \\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY} \end{array} \right\}$ **ONLY.**

A character in a video game is collecting the magic coins that are attached to a cliff face. The character starts at the bottom left corner S and must reach the top right corner E. One point is scored for each coin C that is collected on the way, and the aim is to reach the top right corner with the highest possible score. At each step the character may move either one cell to the right or one cell up, but never down or to the left.

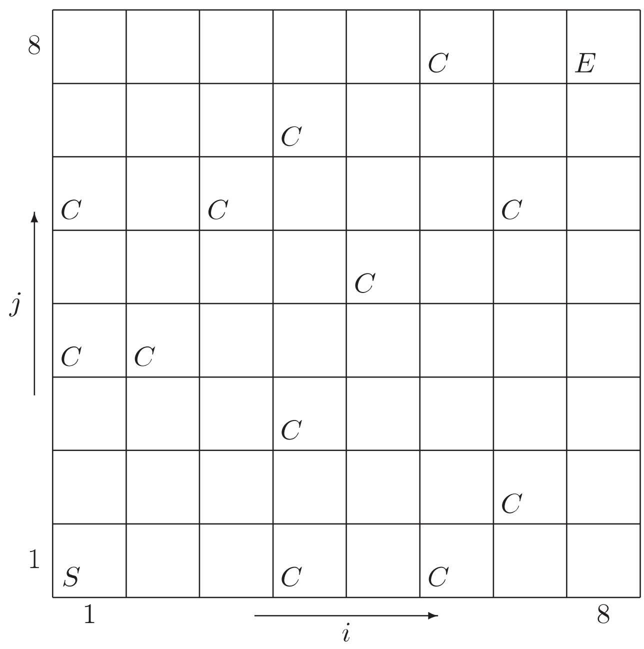


(i) Let $c(i, j) = 1$ if there is a coin at position (i, j), and $c(i, j) = 0$ otherwise. Describe how the maximum score $m(i, j)$ achievable on reaching position (i, j), where $i \geqslant 2$ and $j \geqslant 2$, can be determined in terms of the maximum scores $m(i, j - 1)$ and $m(i - 1, j)$ achievable at the positions immediate below and to the left. Briefly justify your answer.

(ii) Use the result from part (i) to fill in each cell in the diagram above to show the maximum score achievable on reaching that cell. What is the maximum score achievable in the game? *A spare copy of the diagram appears at the end of the question.*

(iii) Given the array of scores $m(i, j)$, describe a method for tracing backwards from E to S a path that, if followed in a forward direction by the character, would achieve the maximum score. Draw one such path across the cliff.

(iv) With the pattern of coins shown, how many different paths from S to E achieve the maximum score? Describe a method for computing the number of such paths.



End of last question

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