UNIVERSITY OF OXFORD

Imperial College London

WARWICK
THE UNIVERSITY OF WARWICK

MATHEMATICS ADMISSIONS TEST

For candidates applying for Mathematics, Computer Science or one of their joint degrees at OXFORD UNIVERSITY and/or IMPERIAL COLLEGE LONDON and/or UNIVERSITY OF WARWICK

Wednesday 30 October 2019

Time Allowed: 2½ hours

Please complete the following details in BLOCK CAPITALS. You must use a pen.

Surname	
Other names	

Candidate Number	**M**					

This paper contains 7 questions of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

A: Oxford Applicants: if you are applying to Oxford for the degree course:

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics, you should attempt Questions **1,2,3,4,5.**
- Mathematics & Computer Science, you should attempt Questions **1,2,3,5,6.**
- Computer Science *or* Computer Science & Philosophy, you should attempt **1,2,5,6,7.**

Directions under A take priority over any directions in B which are relevant to you.

B: Imperial or Warwick Applicants: if you are applying to the University of Warwick for Mathematics BSc, Master of Mathematics, or if you are applying to Imperial College for any of the Mathematics courses**:** Mathematics, Mathematics (Pure Mathematics), Mathematics with a Year in Europe, Mathematics with Applied Mathematics/Mathematical Physics, Mathematics with Mathematical Computation, Mathematics with Statistics, Mathematics with Statistics for Finance, Mathematics Optimisation and Statistics, you should attempt Questions **1,2,3,4,5.**

Further credit cannot be obtained by attempting extra questions. **Calculators are not permitted.**

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

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Wednesday 30 October 2019

Time Allowed: 2½ hours

Please complete these details below in block capitals.

Centre Number															
Candidate Number	**M**														
UCAS Number (if known)				–				–							

	d	d		m	m		y	y
Date of Birth			–			–		

Please tick the appropriate box:

- ☐ I have attempted Questions **1,2,3,4,5**
- ☐ I have attempted Questions **1,2,3,5,6**
- ☐ I have attempted Questions **1,2,5,6,7**

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Q1	Q2	Q3	Q4	Q5	Q6	Q7

1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given **five** possible answers, just one of which is correct. Indicate for each part **A-J** which answer (a), (b), (c), (d), or (e) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)	(e)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

A. The equation

$$x^3 - 300x = 3000$$

has

(a) no real solutions.
(b) exactly one real solution.
(c) exactly two real solutions.
(d) exactly three real solutions.
(e) infinitely many real solutions.

B. The product of a square number and a cube number is

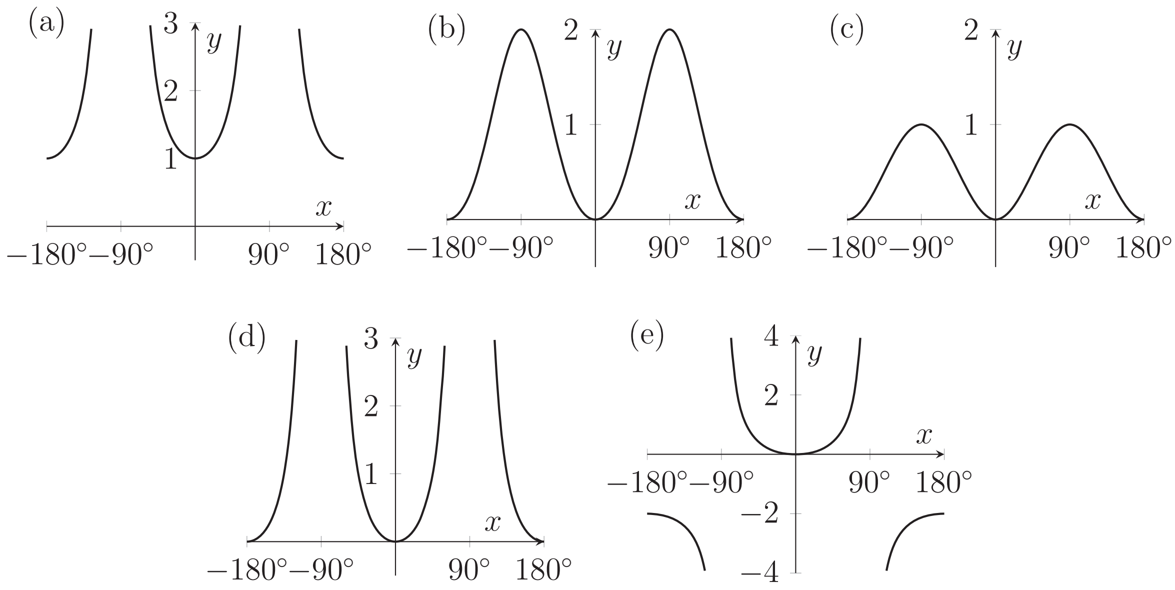
(a) always a square number, and never a cube number.
(b) always a cube number, and never a square number.
(c) sometimes a square number, and sometimes a cube number.
(d) never a square number, and never a cube number.
(e) always a cube number, and always a square number.

Turn over

C. The graph of

$$y = \sin^2 x + \sin^4 x + \sin^6 x + \sin^8 x + \cdots$$

is sketched in



D. The area between the parabolas with equations $y = x^2 + 2ax + a$ and $y = a - x^2$ equals 9. The possible values of a are

(a) $a = 1$, (b) $a = -3$ or $a = 3$, (c) $a = -3$,

(d) $a = -1$ or $a = 1$, (e) $a = 1$ or $a = 3$.

E. The graph of

$$\sin y - \sin x = \cos^2 x - \cos^2 y$$

(a) is empty.
(b) is non-empty but includes no straight lines.
(c) includes precisely one straight line.
(d) includes precisely two straight lines.
(e) includes infinitely many straight lines.

F. In the interval $0 \leqslant x < 360°$, the equation

$$\sin^3 x + \cos^2 x = 0$$

has

(a) 0, (b) 1, (c) 2, (d) 3, (e) 4

solutions.

Turn over

G. Let $a, b, c > 0$. The equations

$$\log_a b = c, \qquad \log_b a = c + \frac{3}{2}, \qquad \log_c a = b,$$

(a) specify a, b and c uniquely.
(b) specify c uniquely but have infinitely many solutions for a and b.
(c) specify c and a uniquely but have infinitely many solutions for b.
(d) specify a and b uniquely but have infinitely many solutions for c.
(e) have no solutions for a, b and c.

H. The triangle ABC is right-angled at B and the side lengths are positive numbers in geometric progression. It follows that $\tan \angle BAC$ is either

(a) $\sqrt{\frac{1+\sqrt{5}}{2}}$ or $\sqrt{\frac{1-\sqrt{5}}{2}}$, (b) $\sqrt{\frac{1+\sqrt{3}}{2}}$ or $\sqrt{\frac{\sqrt{3}-1}{2}}$, (c) $\sqrt{\frac{1+\sqrt{5}}{2}}$ or $\sqrt{\frac{\sqrt{5}-1}{2}}$

(d) $-\sqrt{\frac{1+\sqrt{5}}{2}}$ or $\sqrt{\frac{1+\sqrt{5}}{2}}$, (e) $\sqrt{\frac{1+\sqrt{3}}{2}}$ or $\sqrt{\frac{1-\sqrt{3}}{2}}$.

I. The positive real numbers x and y satisfy $0 < x < y$ and

$$x2^x = y2^y.$$

for
(a) no pairs x and y.
(b) exactly one pair x and y.
(c) exactly two pairs x and y.
(d) exactly four pairs x and y.
(e) infinitely many pairs x and y.

J. An equilateral triangle has centre O and side length 1. A straight line through O intersects the triangle at two distinct points P and Q. The minimum possible length of PQ is

$$\text{(a)}\ \frac{1}{3}, \quad \text{(b)}\ \frac{1}{2}, \quad \text{(c)}\ \frac{\sqrt{3}}{3}, \quad \text{(d)}\ \frac{2}{3}, \quad \text{(e)}\ \frac{\sqrt{3}}{2}.$$

Turn over

2. For **ALL APPLICANTS.**

For k a positive integer, we define the polynomial $p_k(x)$ as

$$p_k(x) = (1+x)(1+x^2)(1+x^3) \times \cdots \times (1+x^k) = a_0 + a_1 x + \cdots + a_N x^N,$$

denoting the coefficients of $p_k(x)$ as $a_0, \ldots, a_N$.

(i) Write down the degree N of $p_k(x)$ in terms of k.

(ii) By setting $x = 1$, or otherwise, explain why

$$a_{\max} \geqslant \frac{2^k}{N+1}$$

where $a_{\max}$ denotes the largest of the coefficients $a_0, \ldots, a_N$.

(iii) Fix $i \geqslant 0$. Explain why the value of a_i eventually becomes constant as k increases.

A student correctly calculates for $k = 6$ that $p_6(x)$ equals

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 4x^7 + 4x^8 + 5x^9 + 5x^{10} + 5x^{11}$$
$$+5x^{12} + 4x^{13} + 4x^{14} + 4x^{15} + 3x^{16} + 2x^{17} + 2x^{18} + x^{19} + x^{20} + x^{21}.$$

(iv) On the basis of this calculation, the student guesses that

$$a_i = a_{N-i} \quad \text{for} \quad 0 \leqslant i \leqslant N.$$

By substituting x^{-1} for x, or otherwise, show that the student's guess is correct for all positive integers k.

(v) On the basis of the same calculation, the student guesses that all whole numbers in the range $1, 2, \ldots, a_{\max}$ appear amongst the coefficients $a_0, \ldots, a_N$, for all positive integers k.

Use part (ii) to show that in this case the student's guess is wrong. Justify your answer.

Turn over
If you require additional space please use the pages at the end of the booklet

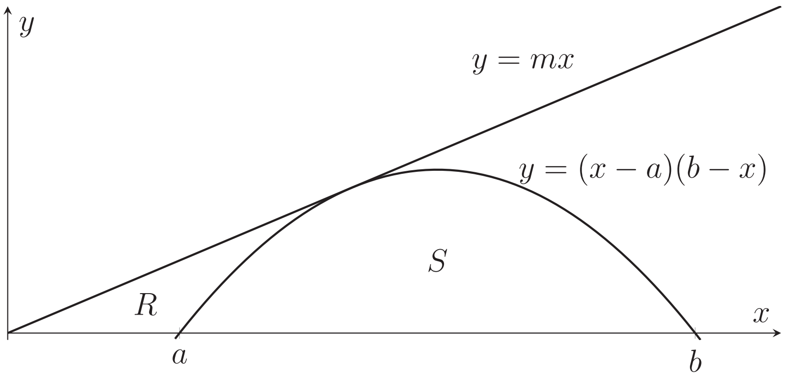
3.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Computer Science and *Computer Science & Philosophy* **applicants should turn to page 14.**

Let a, b, m be positive numbers with $0 < a < b$. In the diagram below are sketched the parabola with equation $y = (x - a)(b - x)$ and the line $y = mx$. The line is tangential to the parabola.

R is the region bounded by the x-axis, the line and the parabola. S is the region bounded by the parabola and the x-axis.



(i) For $c > 0$, evaluate

$$\int_0^c x(c - x)\,\mathrm{d}x.$$

Without further calculation, explain why the area of region S equals $\dfrac{(b-a)^3}{6}$.

(ii) The line $y = mx$ meets the parabola tangentially as drawn in the diagram. Show that $m = \left(\sqrt{b} - \sqrt{a}\right)^2$.

(iii) Assume now that $a = 1$ and write $b = \beta^2$ where $\beta > 1$. Given that the area of R equals $(2\beta + 1)(\beta - 1)^2/6$, show that the areas of regions R and S are equal precisely when

$$(\beta - 1)^2(\beta^4 + 2\beta^3 - 4\beta - 2) = 0. \qquad (*)$$

Explain why there is a solution β to $(*)$ in the range $\beta > 1$.

Without further calculation, deduce that for any $a > 0$ there exists $b > a$ such that the area of region S equals the area of region R.

Turn over
If you require additional space please use the pages at the end of the booklet

4.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Mathematics & Computer Science, *Computer Science* and *Computer Science & Philosophy* **applicants should turn to page 14.**

In this question we will consider subsets S of the xy-plane and points (a, b) which may or may not be in S. We will be interested in those points of S which are nearest to the point (a, b). There may be many such points, a unique such point, or no such point.

(i) Let S be the disc $x^2 + y^2 \leqslant 1$. For a given point (a, b), find the unique point of S which is closest to (a, b).

[You will need to consider separately the cases when $a^2 + b^2 > 1$ and when $a^2 + b^2 \leqslant 1$.]

(ii) Describe (without further justification) an example of a subset S and a point (a, b) such that there is no point of S nearest to (a, b).

(iii) Describe (without further justification) an example of a subset S and a point (a, b) such that there is more that one point of S nearest to (a, b).

(iv) Let S denote the line with equation $y = mx + c$. Obtain an expression for the distance of (a, b) from a general point $(x, mx + c)$ of S.

Show that there is a unique point of S nearest to (a, b).

(v) For some subset S, and for any point (a, b), the nearest point of S to (a, b) is

$$\left(\frac{a + 2b - 2}{5}, \frac{2a + 4b + 1}{5}\right).$$

Describe the subset S.

(vi) Say now that S has the property that

for any two points P and Q in S the line segment PQ is also in S.

Show that, for a given point (a, b), there cannot be two distinct points of S which are nearest to (a, b).

Turn over
If you require additional space please use the pages at the end of the booklet

5. For **ALL APPLICANTS.**

This question is about counting the number of ways of partitioning a set of n elements into subsets, each with at least two and at most n elements. If n and k are integers with $1 \leqslant k \leqslant n$, let $f(n,k)$ be the number of ways of partitioning a set of n elements into k such subsets. For example, $f(5,2) = 10$ because the allowable partitions of $\{1,2,3,4,5\}$ are

$$\begin{array}{ll} \{1,5\},\{2,3,4\}, & \{1,2,5\},\{3,4\}, \\ \{2,5\},\{1,3,4\}, & \{3,4,5\},\{1,2\}, \\ \{3,5\},\{1,2,4\}, & \{1,3,5\},\{2,4\}, \\ \{4,5\},\{1,2,3\}, & \{2,4,5\},\{1,3\}, \\ & \{1,4,5\},\{2,3\}, \\ & \{2,3,5\},\{1,4\}. \end{array}$$

(i) Explain why $f(n,k) = 0$ if $k > n/2$.

(ii) What is the value of $f(n,1)$ and why?

(iii) In forming an allowable partition of $\{1,2,\ldots,n+1\}$ into subsets of at least two elements, we can either

- pair $n+1$ with one other element, leaving $n-1$ elements to deal with, or
- take an allowable partition of $\{1,2,\ldots,n\}$ and add $n+1$ to one of the existing subsets, making a subset of size three or more.

Use this observation to find an equation for $f(n+1,k)$ in terms of $f(n-1,k-1)$ and $f(n,k)$ that holds when $2 \leqslant k < n$.

(iv) Use this equation to compute the value of $f(7,3)$.

(v) Give a formula for $f(2n,n)$ in terms of n when $n \geqslant 1$ and show that it is correct.

Turn over
If you require additional space please use the pages at the end of the booklet

6.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{COMPUTER SCIENCE}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

A flexadecimal number consists of a sequence of digits, with the rule that the rightmost digit must be 0 or 1, the digit to the left of it is 0, 1, or 2, the third digit (counting from the right) must be at most 3, and so on. As usual, we may omit leading digits if they are zero. We write flexadecimal numbers in angle brackets to distinguish them from ordinary, decimal numbers. Thus $\langle 34101\rangle$ is a flexadecimal number, but $\langle 231\rangle$ is not, because the digit 3 is too big for its place. (If flexadecimal numbers get very long, we will need 'digits' with a value more than 9.)

The number 1 is represented by $\langle 1\rangle$ in flexadecimal. To add 1 to a flexadecimal number, work from right to left. If the rightmost digit d_1 is 0, replace it by 1 and finish. Otherwise, replace d_1 by 0 and examine the digit d_2 to its left, appending a zero at the left if needed at any stage. If $d_2 < 2$, then increase it by 1 and finish, but if $d_2 = 2$, then replace it by 0, and again move to the left. The process stops when it reaches a digit that can be increased without becoming too large. Thus, the numbers 1 to 4 are represented as $\langle 1\rangle$, $\langle 10\rangle$, $\langle 11\rangle$, $\langle 20\rangle$.

(i) Write the numbers from 5 to 13 in flexadecimal.

(ii) Describe a workable procedure for converting flexadecimal numbers to decimal, and explain why it works. Demonstrate your procedure by converting $\langle 1221\rangle$ to decimal.

(iii) Describe a workable procedure for converting decimal numbers to flexadecimal, and demonstrate it by converting 255 to flexadecimal.

(iv) We could add flexadecimal numbers by converting them to decimal, adding the decimal numbers and converting the result back again. Describe instead a procedure for addition that works directly on the digits of two flexadecimal numbers, and demonstrate it by performing the addition $\langle 1221\rangle + \langle 201\rangle$.

(v) Given a flexadecimal number, how could you test whether it is a multiple of 3 without converting it to decimal?

(vi) If the $\langle 100000\rangle$ arrangements of the letters *abcdef* are listed in alphabetical order and numbered $\langle 0\rangle$: *abcdef*, $\langle 1\rangle$: *abcdfe*, $\langle 10\rangle$: *abcedf*, etc., what arrangement appears in position $\langle 34101\rangle$ in the list?

Turn over
If you require additional space please use the pages at the end of the booklet

7.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{COMPUTER SCIENCE} \\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY} \end{array} \right\}$ **ONLY.**

You are given two identical black boxes, each with an N-digit display and each with two buttons marked A and B. Button A resets the display to 0, and button B updates the display using a complex, unknown but fixed, function f, so that pressing button A then repeatedly pressing button B displays a fixed sequence

$$x_0 = 0,\ x_1 = f(x_0),\ x_2 = f(x_1),\ \ldots,$$

the same for both boxes. In general $x_i = f^i(0)$ where f^i denotes applying function f repeatedly i times.

You have no pencil and paper, and the display has too many digits for you to remember more than a few displayed values, but you can compare the displays on the two boxes to see if they are equal, and you can count the number of times you press each button.

(i) Explain briefly why there must exist integers i, j with $0 \leqslant i < j$ such that $x_i = x_j$.

(ii) Show that if $x_i = x_j$ then $x_{i+s} = x_{j+s}$ for any $s \geqslant 0$.

(iii) Let m be the smallest number such that x_m appears more than once in the sequence, and let $p > 0$ be the smallest number such that $x_m = x_{m+p}$. Show that if $i \geqslant m$ and $k \geqslant 0$ then $x_{i+kp} = x_i$.

(iv) Given integers i, j with $0 \leqslant i < j$, show that $x_i = x_j$ if and only if $i \geqslant m$ and $j - i$ is a multiple of p. [*Hint*: let r be the remainder on dividing $j - i$ by p, and argue that $r = 0$.]

(v) You conduct an experiment where (after resetting both boxes) you repeatedly press button B once on one box and button B twice on the other box and compare the displays, thus determining the smallest number $u > 0$ such that $x_u = x_{2u}$. What relates the value of u to the (unknown) values of m and p?

(vi) Once u is known, what experiment would you perform to determine the value of m?

(vii) Once u and m are known, what experiment would tell you the value of p?

End of last question

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