# MATHEMATICS ADMISSIONS TEST
# QUESTION BOOKLET

For candidates applying for Mathematics, Computer Science, or one of their joint degrees at the UNIVERSITY OF OXFORD and/or degrees at IMPERIAL COLLEGE LONDON and/or degrees at the UNIVERSITY OF WARWICK.

**October 2023**
**Time allowed: $2\frac{1}{2}$ hours**

You must use a pen throughout the test.
Calculators are not permitted.

**Please write your answers and rough work in the Candidate Answer Booklet.**

During the test, please write your **Candidate Number** at the top of each page of the answer booklet as indicated.

The test contains 6 questions of which you should attempt 5.

If you are applying to Oxford for the degree course:

- Mathematics / Mathematics & Statistics, *or* Mathematics & Philosophy, you should attempt Questions **1, 2, 3, 4, 5**.
- Computer Science, *or* Mathematics & Computer Science, *or* Computer Science & Philosophy, you should attempt Questions **1, 2, 3, 5, 6**.

If you are not an Oxford applicant, you should attempt Questions **1, 2, 3, 4, 5**.

Further credit cannot be obtained by attempting extra questions.

Question 1 is a multiple-choice question with ten parts. Marks are given solely for correct answers, but any rough working should be shown in the pages in the answer booklet. Answer Question 1 on the grid in the answer booklet. Each part is worth 4 marks.

Answers to Questions 2–6 should be written in the space provided in the answer booklet, continuing onto the blank pages at the end of the answer booklet if necessary. Each of Questions 2–6 is worth 15 marks.

**A.** [4 marks] In this question we write $\alpha = \log_{10} 2$, $\beta = \log_{10} 3$, and $\gamma = \log_{10} 7$. Each of the following numbers is close to an integer. Which is the closest to an integer?

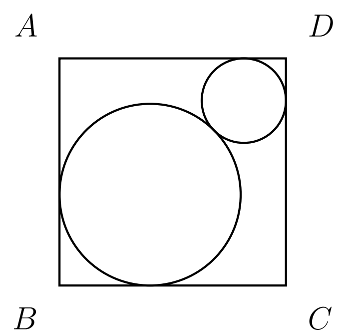
(a) $2\beta$, (b) $5\alpha + \beta$, (c) $\alpha + 2\gamma$, (d) $2\alpha + 5\beta$, (e) $2\alpha + \beta + \gamma$.

[Hint: $2^2 \times 3^5 = 9.72 \times 10^2$]

**B.** [4 marks] Exactly one of these five numbers is a square number. Which one?

(a) 99,999,999, (b) 123,333,333, (c) 649,485,225, (d) 713,291,035, (e) 987,654,000.

**C.** [4 marks] Two circles are inside a square $ABCD$ of side-length 1. One of the circles is tangent to sides $AB$ and to $BC$. The other circle is tangent to sides $CD$ and $DA$. The circles are tangent to each other. The area of the larger circle is 10 times the area of the smaller circle. The diagram is not to scale.




The sum of the radii of the circles is;

(a) $\dfrac{1+\sqrt{10}}{8}$, (b) $2-\sqrt{2}$, (c) $\dfrac{\sqrt{2}}{2}$, (d) $\sqrt{\dfrac{2}{5}}$, (e) $\sqrt{10}-1$.

**D.** [4 marks] How many distinct real solutions $x$ are there to the equation

$$\left(\left(\left(x^2-1\right)^2-2\right)^2-3\right)^2=4 \qquad ?$$

(a) 5, (b) 6, (c) 7, (d) 8, (e) 9.

**E.** [4 marks] The first few positive whole numbers that are not powers of 3 are 2, 4, 5, 6, 7, 8, 10. What is the sum of all the positive whole numbers that are less than $3^{10}$ and are also not powers of 3?

(a) $\dfrac{(3^{10}-1)^2}{2}$, (b) $\dfrac{(3^{11}-1)^2}{2}$, (c) $\dfrac{3(3^{10}-1)^2}{2}$, (d) $(3^{10}-1)^2$, (e) $(3^{11}-1)^2$.

**F.** [4 marks] The coefficient of $x^{12}$ in

$$(1-2x)^5(1+4x^2)^5(1+2x)^5$$

is

(a) $-2^{13}\times 5$, (b) $-2^{12}\times 5$, (c) $2^{12}\times 5$, (d) $2^{13}\times 5$, (e) $0$.

**G.** [4 marks] The real numbers $a$, $b$, and $c$ are non-zero. Each of the following quadratic equations has a repeated real root (not necessarily the same value).

$$ax^2 + bx + c = 0, \qquad bx^2 + cx + a = 0.$$

How many distinct real roots does the equation $cx^2 + ax + b = 0$ have?

(a) 0, (b) 1, (c) 2, (d) 3, (e) It depends on $a$.

**H.** [4 marks] Which of these triangles has the largest area?

(a) an isosceles triangle with side lengths 10, 10, 1;

(b) an isosceles triangle with side lengths 10, 10, 5;

(c) an isosceles triangle with side lengths 10, 10, 10;

(d) an isosceles triangle with side lengths 10, 10, 15;

(e) an isosceles triangle with side lengths 10, 10, 19.

**I.** [4 marks] The polynomial $p(x)$ has degree 3 and has $p(0) = 0$, $p(1) = 1$, $p(2) = 2$. The polynomial has a repeated root at $x = M$ with $M > 0$. The value of $M$ is

(a) $\frac{7}{6}$, (b) $\frac{6}{5}$, (c) $\frac{5}{4}$, (d) $\frac{4}{3}$, (e) $\frac{3}{2}$.

**J.** [4 marks] Let $\lfloor x \rfloor$ denote the largest whole number that is less than or equal to $x$. For example, $\lfloor -\pi \rfloor = -4$. A function $f(x)$ is defined as follows; if $0 < x < 2$ then

$$f(x) = \left(\frac{3}{4}\right)^{\lfloor \log_2(x) \rfloor}$$

and $f(x) = 0$ otherwise. Note that, for example, $f\left(\frac{1}{2}\right) = \frac{4}{3}$. The value of $\int_0^2 f(x)\,\mathrm{d}x$ is

(a) 1, (b) 2, (c) 3, (d) 4, (e) 5.

**2.** For **ALL APPLICANTS.**

For $n$ a positive whole number, and for $x \neq 0$, let $p_n(x) = x^n + x^{-n}$.

(i) [3 marks] Sketch the graph of $y = p_1(x)$. Label any turning points on your sketch.

(ii) [1 mark] Show that $p_2(x) = p_1(x)^2 - 2$.

(iii) [1 mark] Find an expression for $p_3(x)$ in terms of $p_1(x)$.

(iv) [5 marks] Find all real solutions $x$ to the equation

$$x^4 + x^3 - 10x^2 + x + 1 = 0.$$

(v) [5 marks] Find all real solutions $x$ to the equation

$$x^7 + 2x^6 - 5x^5 - 7x^4 + 7x^3 + 5x^2 - 2x - 1 = 0.$$

**3.** For **ALL APPLICANTS.**

Note that the arguments of all trigonometric functions in this question are given in terms of degrees. You are not expected to differentiate such a function. The notation $\cos^n x$ means $(\cos x)^n$ throughout.

(i) [1 mark] Without differentiating, write down the maximum value of $\cos(2x+30°)$.

(ii) [4 marks] Again without differentiating, find the maximum value of

$$\cos(2x+30°)\,(1-\cos(2x+30°))\,.$$

(iii) [4 marks] Hence write down the maximum value of

$$\cos^5(2x+30°)\,(1-\cos(2x+30°))^5\,.$$

(iv) [6 marks] Find the maximum value of

$$\left(1-\cos^2(3x-60°)\right)^4(3-\cos(150°-3x))^8\,.$$

**4.**
For **Oxford applicants** in Mathematics / Mathematics & Statistics / Mathematics & Philosophy, OR those not applying to Oxford, ONLY.

Point $A$ is on the parabola $y = \frac{1}{2}x^2$ at $(a, \frac{1}{2}a^2)$ with $a > 0$. The line $L$ is normal to the parabola at $A$, and point $B$ lies on $L$ such that the distance $|AB|$ is a fixed positive number $d$, with $B$ above and to the left of $A$.

(i) [6 marks] Find the coordinates of $B$ in terms of $a$ and $d$.

(ii) [4 marks] Show that in order for $B$ to lie on the parabola, we must have

$$a^2 d = 2(1 + a^2)^{3/2}. \qquad (*)$$

(iii) [2 marks] Let $t = a^2$ and express the equality $(*)$ in the form $d^{2/3} = f(t)$ for some function $f$ which you should determine explicitly.

(iv) [3 marks] Find the minimum value of $f(t)$. Hence show that the equality $(*)$ holds for some real value of $a$ if and only if $d$ is greater than or equal to some value, which you should identify.

**5.** For **ALL APPLICANTS.**

Define the sequence, $F_n$, as follows: $F_1 = 1$, $F_2 = 1$, and for $n \geqslant 3$,

$$F_n = F_{n-1} + F_{n-2}. \tag{*}$$

(i) [3 marks] What are the values $F_3, F_4, F_5$?

(ii) [1 mark] Using the equation (*) repeatedly, in terms of $n$, how many additions do you need to calculate $F_n$?

We now consider sequences of 0's and 1's of length $n$, that do not have two consecutive 1's. So, for $n = 5$, for example, $(0, 1, 0, 0, 1)$ and $(1, 0, 1, 0, 1)$ would be valid sequences, but $(0, 1, 1, 0, 0)$ would not. Let $S_n$ denote the number of valid sequences of length $n$.

(iii) [1 mark] What are $S_1$ and $S_2$?

(iv) [3 marks] For $n \geqslant 3$, by considering the first element of the sequence of 0's and 1's, show that $S_n$ satisfies the same equation (*). Hence conclude that $S_n = F_{n+2}$ for all $n$.

(v) [2 marks] For $n \geqslant 2$, by considering valid sequences of length $2n - 3$ and focusing on the element in the $(n - 1)^{\text{th}}$ position, show that,

$$F_{2n-1} = F_n^2 + F_{n-1}^2. \tag{O}$$

(vi) [3 marks] For $n \geqslant 2$, show that,

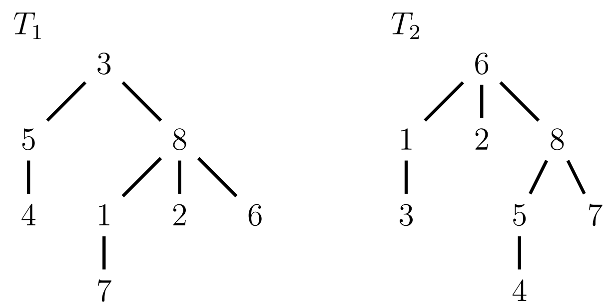
$$F_{2n} = F_n^2 + 2F_nF_{n-1}. \tag{E}$$

(vii) [2 marks] Let $k \geqslant 3$ be an integer. By using the equations (O) and (E) repeatedly, how many arithmetic operations do you need to calculate $F_{2^k}$? You should only count additions and multiplications needed to calculate values using the equations (O) and (E) .

**6.**
For **Oxford applicants** in Computer Science / Mathematics & Computer Science / Computer Science & Philosophy ONLY.

In an *octatree*, *all* the digits 1 to 8 are arranged in a diagram like *trees* $T_1$ and $T_2$ shown below. There is a single digit at the *root*, drawn at the top (so the root is 3 in $T_1$), and every other digit has another digit as its *parent*, so that by moving up the tree from parent to parent, each non-root digit has a unique path to the root. The order in which the children of any parent are drawn does not matter, so for simplicity we show them in increasing order from left to right.

A *leaf* is a digit that is not the parent of any other digit: in tree $T_1$, the leaves are $2, 4, 6$ and 7.




The *code* for an octatree is a sequence of seven digits obtained as follows. We use $T_1$ as an example.

- Remove the numerically smallest leaf and write down its parent. In $T_1$, we remove 2 and write down its parent **8**.
- In the tree that remains, remove the smallest leaf and write down its parent. In $T_1$, after having removed 2, we remove 4 and write down its parent **5**.
- Continue in this way until only the root remains. In $T_1$, we would have deleted the digits $2, 4, 5, 6, 7, 1, 8$ in that order and obtained the code **8538183**.

(i) [1 mark] Find the code for the octatree $T_2$.

(ii) [1 mark] Draw the octatree that has the code **8888888**.

(iii) [2 marks] Draw the octatree that has the code **3165472**.

(iv) [3 marks] What are the leaves of the octatree that has the code **1618388**? Justify your answer.

(v) [2 marks] Find all the digits in the octatree that has the code **1618388** that have 1 as their parent.

(vi) [2 marks] Reconstruct the whole tree that has the code **1618388**.

(vii) [2 marks] Briefly describe a procedure that given a sequence of seven digits from 1 to 8 constructs an octatree with that sequence as its code.

(viii) [2 marks] Is the number of distinct octatrees greater than or smaller than 2,000,000? Justify your answer. (You may use the fact that $2^{10} = 1024$.)