SOLUTIONS FOR ADMISSIONS TEST IN
MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS
THURSDAY 2 NOVEMBER 2017

Mark Scheme:

Each part of Question 1 is worth 4 marks which are awarded solely for the correct answer.

Each of Questions 2-7 is worth 15 marks

QUESTION 1:

A. In order to have two distinct real stationary points we require $4k^2 - 48$ to be positive. Rearranging gives $k > \sqrt{12}$ or $k < -\sqrt{12}$ so **the answer is (b).**

B. Substitute $\gamma = \cos^2(x)$ and solve the resulting quadratic in γ (noting that $0 \leqslant \gamma \leqslant 1$). **The answer is (a).**

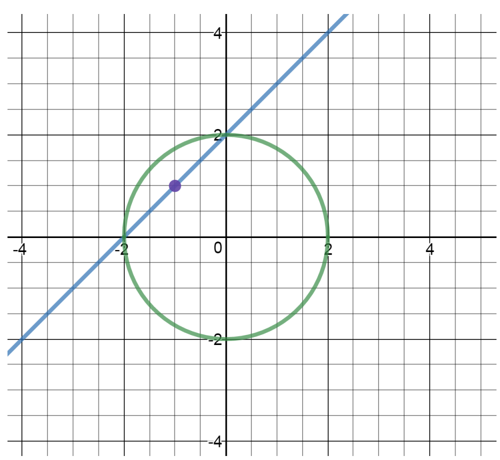
C. Calculating the first few terms gives the pattern $2, 6, 3, \frac{1}{2}, \frac{1}{6}, \frac{1}{3}, 2, 6, \ldots$ The sequence repeats every six terms, and 2017 divided by 6 leaves remainder 1. Hence **the answer is (d).**

D. The graph undergoes two transformations: $y = f(-x)$ is a reflection in the $y-$axis, whilst $y = -f(x)$ is a reflection in the $x-$axis. Combining these two means **the answer is (c).**

E. Rearranging so that $b = 20 - a$ gives $a^2(20 - a)$. Differentiating and setting equal to 0 to find the maximum gives $a = \frac{40}{3}$. The closest integer to this is 13 and so $b = 7$. **The answer is (d).**

F. Rewriting $\tan x = \frac{\sin x}{\cos x}$ gives $\frac{\sin x}{\cos x} < \cos x < \sin x$. In order to satisfy this inequality we know that $\sin x$ cannot be negative, since $\cos x$ would then also be negative, but $\tan x$ would be positive. This rules out (d) and (e). If $0 < \cos x < \sin x$, then $\cos x < \tan x$, so $\cos x \leqslant 0$ and **the answer is (c)**.

G. For $\theta = 0$, the line is simply $y = 1$. Adjusting θ alters the gradient of the line, pivoting around the point $(-1, 1)$ (as shown in the diagram below). By symmetry this shows that there are two possible values to maximise $A(\theta)$ and so **the answer is (b).**



H. By the remainder theorem $f(-b) = 1$, so

$$b^2 + 2ab + a^4 = 1.$$

By the factor theorem $f\left(\frac{1}{a}\right) = 0$,

$$\begin{aligned} \frac{b}{a^2} + \frac{1}{a} + 1 &= 0 \\ b + a + a^2 &= 0 \\ -a - a^2 &= b. \end{aligned}$$

Combining these two equations we find that

$$2a^4 - a^2 = 1$$

and so $a = -1$ or 1. Hence $b = -2$ or 0, and so **the answer is (b)**.

I. By algebraic manipulation we have

$$\begin{aligned} \log_b((b^x)^x) + \log_a\left(\frac{c^x}{b^x}\right) + \log_a\left(\frac{1}{b}\right)\log_a(c) &= 0 \\ x^2 + x(\log_a(c) - \log_a(b)) - \log_a(b)\log_a(c) &= 0 \\ (x + \log_a(c))(x - \log_a(b)) &= 0. \end{aligned}$$

The question asks when the equation has a repeated root, which is satisfied when

$$\begin{aligned} -\log_a(c) &= \log_a(b) \\ \log_a\left(\frac{1}{c}\right) &= \log_a(b) \end{aligned}$$

and so **the answer is (d).**

J. Considering each of the integrals in turn:

(a) note that $(x^2 - 4)\sin^8(\pi x) \leqslant 0$ when $0 \leqslant x \leqslant 2$ so this integral will evaluate as negative;

(b) $(2 + \cos x)$ will always evaluate positive as it translates the $y = \cos x$ graph positively along the y-axis, so the integral will be positive and will be less than 54π for $0 \leqslant x \leqslant 2\pi$;

(c) the maximum value of $\sin^{100}(x)$ is 1, and so the value of the integral for $0 \leqslant x \leqslant \pi$ must be $\leqslant \pi$;

(d) for $0 \leqslant x \leqslant \pi$ the smallest $(3 - \sin x)^6$ can be is 64π;

(e) $(\sin^3(x) - 1)$ will always be $\leqslant 0$, and so this integral will evaluate as negative.

The answer is (d).

2. (i) [3 marks] We have $\alpha^2(1+\alpha)=1$. Now:
if $\alpha \leqslant -1$ then $\alpha^2(1+\alpha) \leqslant 0$.
if $-1<\alpha<0$ then $\alpha^2(1+\alpha)<\alpha^2<1$.
if $\alpha \geqslant 1$ then $\alpha^2(1+\alpha) \geqslant 2$.
Hence $0<\alpha<1$ is the only possibility.

[Alternative approach: students might sketch $y=x^3+x^2$ noting that the local maximum in $-1<x<0$ is at $x=-2/3$ where $y=4/27$.]

(ii) [2 marks] Multiplying the defining equation for α by α and rearranging we have

$$\alpha^4=\alpha-\alpha^3=\alpha-(1-\alpha^2)=-1+\alpha+\alpha^2.$$

(iii)(a) [1 mark] Dividing the defining equation by α we find

$$\frac{1}{\alpha}=\alpha+\alpha^2.$$

(So $A=0,\ B=1,\ C=1$.)

(b) [2 marks] The sum is a geometric series with sum $(1+\alpha)^{-1}$ and dividing the defining equation by $1+\alpha$ we find

$$\frac{1}{1+\alpha}=\alpha^2.$$

(So $A=0,\ B=0,\ C=1$.)

(c) [4 marks] From the defining equation we have $1-\alpha^3=\alpha^2$ and so by the difference of two cubes we know that

$$\alpha^2=(\alpha^2+1+\alpha)(1-\alpha)$$

and then by (iii)(a) and (ii)

$$\begin{aligned}
\frac{1}{1-\alpha} &= \frac{\alpha^2+\alpha+1}{\alpha^2}\\
&= 1+\frac{1}{\alpha}+\frac{1}{\alpha^2}\\
&= 1+\left(\alpha+\alpha^2\right)+(\alpha+\alpha^2)^2\\
&= 1+\alpha+\alpha^2+\alpha^2+2(1-\alpha^2)+(-1+\alpha+\alpha^2)\\
&= 2+2\alpha+\alpha^2.
\end{aligned}$$

(So $A=2,\ B=2,\ C=1$.)

[Alternative method: as uniqueness of the expression is given we can write $(1-\alpha)^{-1}=A+B\alpha+C\alpha^2$ and so

$$\begin{aligned}
1 &= (A+B\alpha+C\alpha^2)(1-\alpha)\\
&= A+B\alpha+C\alpha^2-A\alpha-B\alpha^2-C(1-\alpha^2)\\
&= (A-C)+(B-A)\alpha+(2C-B)\alpha^2.
\end{aligned}$$

By uniqueness we can compare coefficients to find

$$A-C=1, \qquad B-A=0, \qquad 2C-B=0,$$

and solving these equations gives $A = 2,\ B = 2, C = 1$.]

[Alternative method 2: we can write

$$(1-\alpha)(1+\alpha+\alpha^2+\alpha^3+\ldots) = 1.$$

From the defining equation we know that

$$(1-\alpha)(1+\alpha+1+\alpha^2+\alpha^4+\ldots) = 1.$$

So we have another geometric series with sum $(1-\alpha^2)$. Hence

$$\begin{aligned} 1 &= (1-\alpha)(1+\alpha+1+\alpha^2+\alpha^4+\ldots) \\ 1 &= (1-\alpha)(1+\alpha+\frac{1}{1-\alpha^2}) \end{aligned}$$

And rearranging gives the answer.]

(d) [3 marks] As every natural number k can be written uniquely in binary then every power α^k appears once and once only in the infinite product and so

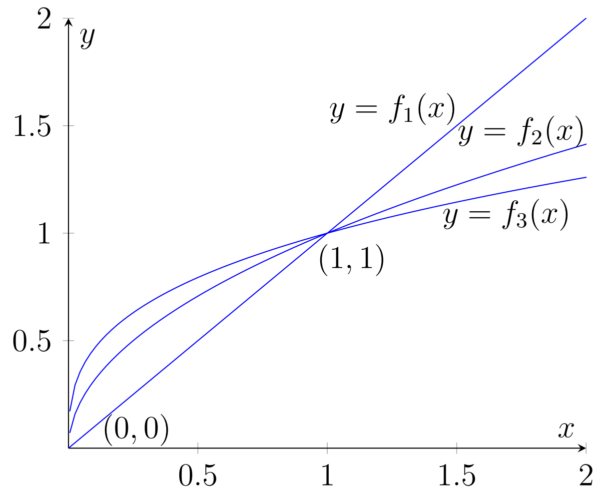
$$\begin{aligned} & (1+\alpha)(1+\alpha^2)(1+\alpha^4)(1+\alpha^8)(1+\alpha^{16})\cdots \\ = \;& 1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\cdots \\ = \;& \frac{1}{1-\alpha} \\ = \;& 2+2\alpha+\alpha^2 \end{aligned}$$

from (iii)(c). (So $A = 2,\ B = 2,\ C = 1$.) A rigorous explanation for the infinite product equalling the infinite sum should not be expected and indeed simple pattern spotting of the form

$$1+\alpha = 1+\alpha; \qquad (1+\alpha)(1+\alpha^2) = 1+\alpha+\alpha^2+\alpha^3; \qquad (1+\alpha)(1+\alpha^2)(1+\alpha^4) = 1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6+\alpha^7,$$

should be considered sufficient.

3. (i) [3 marks]



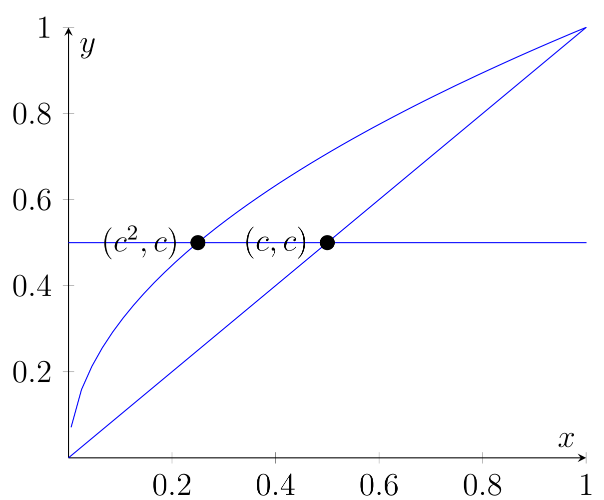
(ii) [5 marks] Region between f_k and f_{k+1} has area

$$\begin{aligned}
\int_0^1 x^{\frac{1}{k+1}} - x^{\frac{1}{k}} &= \left[\frac{k+1}{k+2}x^{\frac{k+2}{k+1}} - \frac{k}{k+1}x^{\frac{k+1}{k}}\right]_0^1 \\
&= \frac{k+1}{k+2} - \frac{k}{k+1} \\
&= \frac{1}{(k+1)(k+2)}
\end{aligned}$$

Hence the region between f_1 and f_2 has area $\frac{1}{2\times 3} = \frac{1}{6}$.

(iii) [1 mark] $x^{\frac{1}{2}} = c$, so x-coordinate of point of intersection between $y = c$ and $f_2(x)$ is c^2. x-coordinate of point of intersection between $y = c$ and $f_1(x)$ is c.

(iv) [6 marks]



Area beneath $f_2(x)$ between 0 and c^2 is $\int_0^{c^2} x^{\frac{1}{2}}\mathrm{d}x$.
Rectangle area is equal to $c(c - c^2)$.

Area beneath $f_1(x)$ between 0 and c equal to $\frac{1}{2}c^2$.
Hence

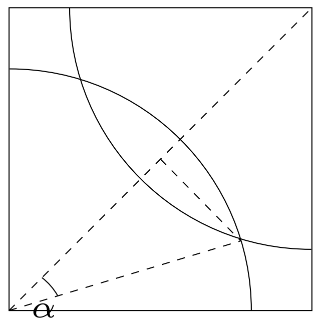
$$\begin{aligned}
\frac{1}{12} &= \int_0^{c^2} x^{\frac{1}{2}}\mathrm{d}x + c(c-c^2) - \frac{1}{2}c^2 \\
&= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^{c^2} + c(c-c^2) - \frac{1}{2}c^2 \\
&= \frac{2}{3}c^3 + c^2 - c^3 - \frac{1}{2}c^2 \\
&= -\frac{1}{3}c^3 + \frac{1}{2}c^2
\end{aligned}$$

And so $4c^3 - 6c^2 + 1 = 0$. By inspection or otherwise, $c = \frac{1}{2}$.

4.(i) [2 marks] We require that the area of the sector is $\frac{r^2\theta}{2} = \frac{1}{2}$. $\theta = \frac{\pi}{2}$ as a quarter circle is formed. Hence the length of rope required is $\sqrt{\frac{2}{\pi}}$.

(ii) [2 marks] Both horses can reach half the area of the field, so without overlap these would sum to 1. This is the same area as the total field, hence the overlapping area between the two quarter circles must equal the areas outside the quarter circles.

(iii) [5 marks]



To calculate α: Distance from bottom left corner of square to centre is $\frac{\sqrt{2}}{2}$. Hypotenuse (distance from bottom left corner to point where both quarter circles intersect) is the radius calculated in part (i), so $\sqrt{\frac{2}{\pi}}$. So $\cos(\alpha) = \frac{\frac{\sqrt{2}}{2}}{\sqrt{\frac{2}{\pi}}} = \frac{\sqrt{\pi}}{2}$. Hence $\alpha = \cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right)$.

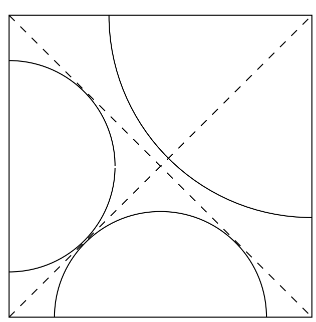
To calculate overlapping area: Length of line from centre of field to intersection of quarter circles $= \sqrt{\left(\sqrt{\frac{2}{\pi}}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{4-\pi}{2\pi}}$

Area of triangle $= \frac{\frac{\sqrt{2}}{2}\sqrt{\frac{4-\pi}{2\pi}}}{2} = \sqrt{\frac{4-\pi}{16\pi}}$

Area of sector $= \frac{\frac{2}{\pi}\cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right)}{2} = \frac{1}{\pi}\cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right)$

So total area $= \frac{4}{\pi}\cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right) - \sqrt{\frac{4-\pi}{\pi}}$ as required.

(iv) [4 marks]



We require all the areas to be equal. Two distinct shapes will be formed - two half circles and one quarter circle. Hence we require $\frac{\pi r^2}{2} = \frac{\pi s^2}{4}$, where r is the radius of the half circles and s is the radius of the semi-circles. Hence we know the relationship between r and s is $2r^2 = s^2$.
To calculate r note that both half circles can expand to touch on the $y = x$ diagonal. As the horses are attached at the midpoint of the fence, the diagonal distance between midpoints must be $\frac{1}{\sqrt{2}}$.
Hence $r = \frac{1}{2\sqrt{2}}$.
Using the relationship above we know that $s = \frac{1}{2}$.

(v) [2 marks] From previous part note that $0 \leqslant g \leqslant \frac{1}{2\sqrt{2}}$. We can relate g to h as $(g+h)^2 = 0.5^2 + 1^2$ by Pythagoras, so $g = \sqrt{\frac{5}{4}} - h$. As we know the maximum h can be is 1, the actual range for g is $\frac{\sqrt{5}-2}{2} \leqslant g \leqslant \frac{1}{2\sqrt{2}}$.

5. (i) [2 marks] Each time the teacher gives out a sweet, the number of children skipped increases by 1, so the kth sweet will be given after $\frac{k(k+1)}{2}$ steps. However, as there are 10 children in the circle, we only need to keep track of the unit digit, since every 10 steps the unit digit will return to the original unit digit.

(ii) [3 marks]

$$\frac{(20-k-1)(20-k)}{2} = \left(\frac{k(k+1)}{2} + 10(20-2k-1)\right)$$

So they must have the same final digits, as we're adding something that will always be a multiple of 10, so adding 0 from the units digit.
$\frac{(20-k-1)(20-k)}{2}$ is the formula for the number of steps until the $(20-k-1)$th sweet is given out, and so both the kth and $(20-k-1)$th sweet must go to the same child.

(iii) [3 marks] The 18th sweet is given to the same child who received the 1st sweet, c_1. The 19th and 20th sweets are given to child c_0. Hence after 20 sweets have been given out the teacher returns to the first child (for the 21st sweet), and the pattern will repeat.

(iv) [2 marks] As the pattern repeats with period 20, and sweets 10-18 are given to the same children as 1-9. Children 2, 4, 7 and 9 never receive any sweets.

(v) [2 marks] If there were n sweets initially, we must have $\frac{n(n+1)}{2} = 1830$. So $n = 60$ (by inspection, or solving a quadratic equation.)

(vi) [3 marks] At the end of each period, 20 sweets are given out, and the maximum number of sweets a child can get is 4; this maximum is received by children 0, 1, 5 and 6. Thus after $60/20 = 3$ periods, the maximum a child can receive is 12 (received by the above four children).

6. (i) [2 marks] All the other orderings of apples, bread, and carrots are unsafe.

(ii) [2 marks] Here's one answer.

Item	w_i	s_i
Beer	5	10
Camembert	7	3

This is unsafe; but swapping the order gives a safe packing.

More generally, a solution with two objects and $s_2 < w_1 < w_2 \leq s_1$ will be correct. There are also solutions with more than two objects.

(iii) [2 marks] Here's one answer.

Item	w_i	s_i
Wine	10	4
Eggs	2	6

Again this is unsafe; but swapping the order gives a safe packing.

More generally, a solution with two objects with $w_2 \leq s_1 < s_2 < w_1$ will be correct. There are also solutions with more than two objects.

(iv) [5 marks] Let w be the weight of all the items above item j. Since the packing is safe, we have

$$\begin{align} w &\leq s_j \tag{1} \\ w + w_j &\leq s_i \tag{2} \end{align}$$

When we swap the items, item i now has weight w above it. From inequality (2)[1], $w \leq s_i$, so item i is still not squashed.

When we swap the items, item j now has weight $w + w_i$ above it. But

$$w + w_i \leq w + w_j - s_i + s_j \leq s_j$$

using the inequality given in the question and (2). So item j is still not squashed.

(v) [4 marks] Arrange the items by order of weight + strength (largest value at the bottom).

Note that the inequality given in question (iv) is equivalent to

$$w_j + s_j \geq w_i + s_i.$$

Suppose we have a safe packing order, but that two consecutive items are not in weight+strength order. By part (iv), if we swap these two items, the order will still be safe.

We can repeat this operation, continually swapping adjacent items, until the whole packing is in weight+strength order. The resulting order is still safe.

[1] Strictly speaking, we also require that $w_j \geq 0$; it is not necessary to state this explicitly.

7. (i) [2 marks] $R(a+b) = R(b) + R(a)$ and $R(R(a)) = a$

(ii) [2 marks] $S_k(a+b) = b + R(a)$. Then

$$S_k(S_k(a+b)) = S_k(b + R(a)) = R(a) + R(b)$$

(iii) [4 marks]

$$\begin{aligned} S_5(1,2,3,4,5,6,7,8) &= (6,7,8,5,4,3,2,1). \\ S_5(6,7,8,5,4,3,2,1) &= (3,2,1,4,5,8,7,6) = (3,2,1) + (4,5) + (8,7,6) \end{aligned}$$

Then

$$\begin{aligned} S_5((3,2,1) + (4,5) + (8,7,6)) &= (8,7,6) + (5,4) + (1,2,3) \\ S_5((8,7,6) + (5,4) + (1,2,3)) &= (1,2,3) + (4,5) + (6,7,8). \end{aligned}$$

(iv) [4 marks] Now let a be a sequence of length n with $k \geqslant n/2$. We will write $a = x + y + z$ where x is the first $n-k$ elements, y is the next $2k-n$ and z is the final $n-k$ elements. Then

$$\begin{aligned} S_k(x+y+z) &= z + R(x+y) = z + R(y) + R(x) \\ S_k(z + R(y) + R(x)) &= R(x) + R(z + R(y)) = R(x) + y + R(z) \\ S_k(R(x) + y + R(z)) &= R(z) + R(R(x) + y) = R(z) + R(y) + x \\ S_k(R(z) + R(y) + x) &= x + R(R(z) + R(y)) = x + y + z \end{aligned}$$

(v) [3 marks] Take $a = (1,2,3,4,5)$ and $k = 2$. We see that

$$(1,2,3,4,5) \xrightarrow{S_2} (3,4,5,2,1) \xrightarrow{S_2} (5,2,1,4,3) \xrightarrow{S_2} (1,4,3,2,5) \xrightarrow{S_2} (3,2,5,4,1)$$

which is not in its original order but with two further applications we get

$$(3,2,5,4,1) \xrightarrow{S_2} (5,4,1,2,3) \xrightarrow{S_2} (1,2,3,4,5)$$

we get the original sequence, so 6 performances in all.