SOLUTIONS FOR ADMISSIONS TEST IN
MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS
WEDNESDAY 04 NOVEMBER 2020

Mark Scheme:
Each part of Question 1 is worth 4 marks which are awarded solely for the correct answer.
Each of Questions 2–7 is worth 15 marks

1

A The centre is at $(3, 4)$ and one corner is at $(1, 5)$, which is a displacement of $(-2, 1)$. There's another corner opposite, at $(3, 4) - (-2, 1) = (5, 3)$, but this is not one of the options. To find the other corners, we need to find a vector with equal magnitude, but at right-angles to $(-2, 1)$. Rotating by $90°$ gives $(1, 2)$. The other corners are at $(3, 4) \pm (1, 2)$, which includes $(2, 2)$
The answer is (d)

B Using the difference of two squares, this is $\int_0^1 e^{2x} - x^2 \, \mathrm{d}x$. Notice that, since the derivative of e^{2x} is $2e^{2x}$, this integral is

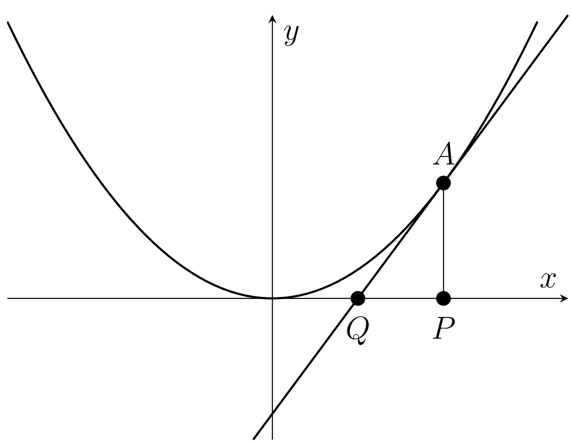
$$\left[\frac{1}{2}e^{2x} - \frac{x^3}{3}\right]_0^1 = \frac{e^2}{2} - \frac{1}{3} - \frac{1}{2} = \frac{3e^2 - 5}{6}$$

The answer is (d)

C Pair the terms to get $-3 - 7 - \cdots - 199$. This is the sum of an arithmetic progression and is equal to $(-3 - 199) \times 50/2 = -5050$
The answer is (e)

D $3\cos^2 x + 2\sin x + 1 = 3(1 - \sin^2 x) + 2\sin x + 1 = -3(\sin x - 1/3)^2 + 1/3 + 4$. This takes its maximum value when $\sin x = 1/3$, and that value is $4 + 1/3 = 13/3$.
The answer is (b)

E The tangent is $y = 2ax - a^2$, which crosses the x-axis at $(a/2, 0)$.



The area is $\int_0^a x^2 \, \mathrm{d}x -$ (area of APQ) where A is (a, a^2), P is $(a, 0)$ and Q is $(a/2, 0)$. This is $\dfrac{a^3}{3} - \dfrac{a^3}{4} = \dfrac{a^3}{12}$
The answer is (c)

F Factorise the numbers from 1 to 10; this gives

$$\log_{10}(2 \times 5 \times 3^2 \times 2^3 \times 7 \times 2 \times 3 \times 5 \times 2^2 \times 3 \times 2) = \log_{10}(2^8 3^4 5^2 7).$$

Use $\log_{10}(2^2 5^2) = 2$. Use $2^4 3^4 = 6^4$. This gives $2+\log_{10}(2^2 6^4 7) = 2+2\log_{10} 2+4\log_{10} 6+\log_{10} 7$.
The answer is (c)

G The turning points have $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at $x = 1$ and at $x = 3$, so $3 + 2a + b = 0$ and $27 + 6a + b = 0$. Solve these for $a = -6$, $b = 9$. Now $y = 2$ at $x = 1$ so $c = -2$. Then $y = -2$ at $x = 3$ gives $d = -2$.
The answer is (b)

H Option (a) looks like a quartic, options (b), (d), and (e) look like cubics, and option (c) looks like a quadratic, so we might expect that one of the cubics is the derivative of option (a), and option (c) is the derivative of another of the cubics, leaving one remaining cubic to be $h(x)$.

Option (c) has the right sign, and zeros in the right places, to be the derivative of option (b). Also, option (e) has the right number of zeros to be the derivative of option (a). That leaves option (d) as a possible candidate for $h(x)$.

We should check that it's definitely not the derivative of any of the other options, and that its derivative is not any of the other options. It's not the derivative of (a) or (c) or (e) because it's got the wrong number of zeros, and it's the not the derivative of (c) because it's negative for large x. It's also not the case that the derivative of option (d) is any of the other graphs; such a graph would have two zeros for the two turning points of (d), but only (c) has two zeros, and the sign of option (c) is wrong (positive where the gradient of option (d) is negative). So we conclude that (c) is the derivative of (b), (e) is the derivative of (a) and option (d) is the "odd one out"; it is $h(x)$.
The answer is (d)

I The geometric sum converges if $\frac{1}{|\tan x|} < 1$, and converges to

$$\frac{1}{\tan x} + \frac{1}{\tan^2 x} + \frac{1}{\tan^3 x} + \cdots = \frac{\frac{1}{\tan x}}{1 - \frac{1}{\tan x}} = \frac{1}{\tan x - 1}.$$

Let $u = \tan x$, then this sum is equal to $\tan x$ if and only if $u^2 - u = 1$, so if and only if $u = \frac{1}{2}(1 \pm \sqrt{5})$. But the condition $|\tan x| > 1$ means that we must take $u = \frac{1}{2}(1 + \sqrt{5})$. There is one value of x in the range $-90° < x < 90°$ with this value of $\tan x$.
The answer is (b)

J For $r < 1$, $A = 0$ and $B = 4 - \pi r^2$. For $r > \sqrt{2}$, $A = \pi r^2 - 4$ and $B = 0$, which eliminates options (a) and (c). Now options (b) and (e) both have $A + B = 0$ at some value of r, which would require $A = 0$ and $B = 0$, that is; no area inside the square that's not also inside the circle and vice versa. This is impossible, and the only remaining option is (d).
The answer is (d)

2

Different alternative solutions are indicated with (Alt1), (Alt2), and so on.

(i) $100 = 64 + 32 + 4 = 2^6 + 2^5 + 2^2$, so 100 is 1100100_2 in binary. **2 marks**

(ii) $gfgf(1) = 100$ **2 marks**

(iii) (Alt1) 200 would have had to come from 50 since 200 is not odd, and 50 can't have come from f (not odd) or from g (not multiple of 4)

(Alt2) In binary, f adds 1 to the end, g adds 00 to the end. 200 ends in 000, which we can't have. **2 marks**

(iv) (Alt1) Work backwards; if number is odd, must have come from f, and if multiple of 4, must have come from g.

(Alt2) In binary, read from the left. Must be sequence of 00s and 1s. **2 marks**

(v) An element of S in the range $2^{k+2} \leqslant n < 2^{k+3}$ can end in 00 after an element of S in the range $2^k \leqslant n < 2^{k+1}$ (one of u_k) or can end in 1 after an element of S in the range $2^{k+1} \leqslant n < 2^{k+2}$ (one of u_{k+1}). We should add these possibilities. **3 marks**

(vi) (Alt1) Note that $s_k = u_0 + u_1 + \cdots + u_{k-1} + u_k$.

If we sum the recursion relation in (v) over $0, 1, 2, ..., k$ we get $(s_{k+2} - u_1 - u_0) = (s_{k+1} - u_0) + s_k$.

As $u_0 = u_1 = 1$ (because 1_2 and 11_2 are the only such n) then this rearranges to $s_{k+2} = s_{k+1} + s_k + 1$.

(Alt2) Elements in the range for s_{k+2} are either odd, and come from applying f to an element of s_{k+1}, or are even, and come from applying g to an element of s_k.

The exception is 1, which should not be included in either of these cases.

So $s_{k+2} - 1 = s_{k+1} + s_k$

(Alt3) Proof by induction;

First note that $s_{k+1} = s_k + u_{k+1}$.

If $s_{k+2} = s_{k+1} + s_k + 1$ then $s_{k+3} = u_{k+3} + s_{k+2} = u_{k+3} + s_{k+1} + s_k + 1$. Now use (v) for

$$s_{k+3} = u_{k+2} + u_{k+1} + s_{k+1} + s_k + 1$$

Now re-combine;

$$s_{k+3} = (s_{k+1} + u_{k+2}) + (s_k + u_{k+1}) = s_{k+2} + s_{k+1} + 1.$$

Check the base case $s_2 = s_1 + s_0 + 1$ and conclude. **4 marks**

3

(i) When $y = 0$ then $x = -1$, 0, or 1. Hence $a = -1$ and $b = 1$. **2 marks**

(ii) Complete the square $\left(y - \frac{1}{2}\right)^2 - \frac{1}{4} = x^3 - x$. So for there to be just one value possible value of y for a given x then we need $y = \frac{1}{2}$. So $\delta = \frac{1}{2}$. **2 marks**

(iii) We have $y = \frac{1}{2} \pm \sqrt{x^3 - x + \frac{1}{4}}$

If $(x, \frac{1}{2} + Y)$ is on the curve then so is $(x, \frac{1}{2} - Y)$. **2 marks**

(iv) We then see that corresponding values of x must satisfy $x^3 - x + \frac{1}{4} = 0$, and α, β, γ are roots of this cubic. **1 mark**

(v) By the factor theorem, and considering the coefficient of x^3, this cubic must factorise as $x^3 - x + \frac{1}{4} = (x - \alpha)(x - \beta)(x - \gamma)$. When expanding the RHS, the coefficient of x^2 is $-(\alpha + \beta + \gamma)$ and so, comparing the x^2 coefficient, we have $\alpha + \beta + \gamma = 0$. **3 marks**

(vi) The equation of C is given by

$$\left(x - \frac{\alpha + \beta}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{\beta - \alpha}{2}\right)^2$$

and so C and S intersect when

$$\frac{1}{4} + x^3 - x - \left(\frac{\beta - \alpha}{2}\right)^2 + \left(x - \frac{\alpha + \beta}{2}\right)^2 = 0.$$

Note that α, β already satisfy this. If the third root is ζ then we have $\alpha + \beta + \zeta = -1$ (as the coefficient of x^2 is 1, based on the same argument as in (v)). Hence C and S meet at two further points with x-co-ordinate $\zeta = \gamma - 1$. **5 marks**

4

Different alternative solutions are indicated with (Alt1), (Alt2), and so on.

Alternative solutions marked with an asterisk (*) have been added to give marks to candidates misled by the typographical error in the question.

(i) (a) (Alt1) The graph of an even function has reflective symmetry in the y-axis.

The graph of an odd function has rotational symmetry of order 2 about the origin.

(Alt2*) The graph of an even function has reflective symmetry in the y-axis.

The graph of an odd function (as defined in the question) has reflectional symmetry in the x-axis (or you could say that it has reflectional symmetry in the y-axis). **2 marks**

(b) (Alt1) The tangent line at the point $(x, f(x))$ has gradient $f'(x)$. Reflecting that line in the y-axis gives a line of gradient $-f'(x)$ so that $f'(-x) = -f'(x)$ when f is even.

Rotating by a half turn doesn't change the gradient so that $f'(-x) = f'(x)$ when f is odd.

(Alt2*) With the definition of an odd function printed in the question, if f is odd then $f(x) = 0$ for all x, so $f'(x) = 0$ for all x, so $f'(-x) = f'(x)$ and f' is even. **3 marks**

(ii) (a) Area $A(\theta)$ is the triangle below the line, plus a triangle with angle θ above the line.

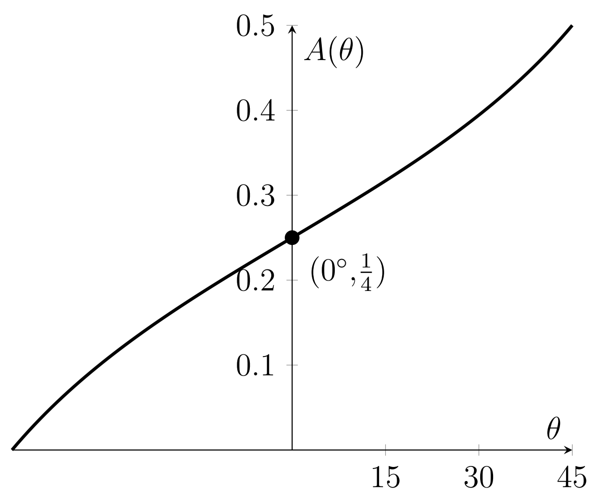
Area $A(-\theta)$ is the triangle below the line, minus a triangle with angle θ below the line.

Adding gives twice the area of the triangle under the line for total $\frac{1}{2}$. **2 marks**

(b) For $0 < \theta < 45°$ the further triangle has base $1/\sqrt{2}$ and has height $\tan\theta/\sqrt{2}$

So the total area of $A(\theta)$ equals $(1 + \tan\theta)/4$ **2 marks**

(c)



2 marks

(d) (Alt1) Rewrite previous expression as $A(\theta) - \frac{1}{4} = \frac{1}{4} - A(-\theta)$ so $A(\theta) - \frac{1}{4}$ is odd and so graph has rotational symmetry of order 2 about the point $\left(0, \frac{1}{4}\right)$.

(Alt2) Note that $\tan(\theta) = -\tan(-\theta)$; this is symmetry under two reflections, which is a rotation. So this has rotational symmetry. **2 marks**

(e) (Alt1) $A(\theta) - 1/4$ is odd, so $A''(\theta)$ is odd by (i)(b). So $A''(0) = -A''(-0)$ must be 0.

(Alt2) If the gradient is decreasing on the left, then by rotational symmetry it is increasing on the right, so goes through a point with second derivative zero at $\theta = 0$. **2 marks**

5

(i) Miriam eats the greatest number in total if the sunny days come first: in that case, she eats i sweets on the i'th day for the first 15 days, making $\frac{1}{2}15(15+1)$ sweets, and then 15 sweets on each of the 15 remaining days, making a total of 345 sweets.

She eats the smallest number in total if the n sunny days come last: then the total is just $\frac{1}{2}n(n+1)$. For $n = 15$, that's 120 sweets. **4 marks**

(ii) If the sunny days come first, Adam eats $15 + i$ sweets on day $15 + i$ for i from 1 up to 15, a total of $15^2 + \frac{1}{2}15(15+1)$ which is 345 sweets.

If the sunny days come last, then he eats i sweets on day i for the first 15 days, a total of $\frac{1}{2}15(15+1) = 120$. This is the same as Miriam in each case. **3 marks**

(iii) If rainy day k is swapped with sunny day $k+1$, and Miriam would have eaten a and $a+1$ on the two days, she now eats $a+1$ and $a+1$.

The number of sweets eaten by each on preceding and on subsequent days does not change.

This is an increase of one sweet.

Adam, on the other hand, eats $k+1$ sweets on the second day instead of k on the first, so his intake increases by 1 also. **4 marks**

(iv) Any arrangement with equal numbers of sunny and rainy days can be reached by starting with all the rainy days first, and repeatedly swapping a rainy day with a sunny day immediately after it. Specifically, we can swap the last rainy day with successive sunny days until it is in the desired place, then move the penultimate rainy day, and so on.

Since Miriam and Adam start this process eating the same total number of sweets, and their totals both increase by 1 on each swap, they end up eating the same number of sweets also.

4 marks

6

Different alternative solutions are indicated with (Alt1), (Alt2), and so on.

(i) $g(1,k) = k$ for $k \geq 1$. **1 mark**

(ii) $g(n,1) = 1$. **1 mark**

(iii) $g(n,k) = g(n-1,k) + g(n,k-1)$. **4 marks**

(iv) (Alt1) Tabulating as shown, we deduce $g(7,5) = 330$.

n	$g(n,1)$	$g(n,2)$	$g(n,3)$	$g(n,4)$	$g(n,5)$
1	1	2	3	4	5
2	1	3	6	10	15
3	1	4	10	20	35
4	1	5	15	35	70
5	1	6	21	56	126
6	1	7	28	84	210
7	1	8	36	120	330

(Alt2) Line up seven tennis balls and four sticks in a row. Any arrangement gives a way to split the tennis balls into five groups (one for each stocking), divided by the sticks. There are $\binom{11}{4} = 330$ ways to arrange seven tennis balls and four sticks in a row. **4 marks**

(v) (Alt1) Santa can put one ball in each stocking, then distribute the remaining 2 balls however he chooses, and there are $g(2,5) = 15$ ways to do this. (In general $h(n,k) = g(n-k,k)$.)

(Alt2) Observe that $h(n,n) = 1$ and $h(n,1) = 1$. For $n \geq 2$ and $2 \leq k < n$, Santa must give the first ball to the first child, then he can distribute the remaining balls among all the children in the same way, or he can give no more to the first child and distribute them among the others: so $h(n,k) = h(n-1,k) + h(n-1,k-1)$. We can tabulate to find $h(7,5) = 15$.

n	$h(n,1)$	$h(n,2)$	$h(n,3)$	$h(n,4)$	$h(n,5)$
1	1				
2	1	1			
3	1	2	1		
4	1	3	3	1	
5	1	4	6	4	1
6	1	5	10	10	5
7	1	6	15	20	15

(Alt3) Put five tennis balls into the stockings. Now line up two tennis balls and four sticks in a row. The four sticks split the tennis balls into five groups (one for each stocking), divided by the sticks. There are $\binom{6}{2} = 15$ ways to arrange two tennis balls and four sticks in a row.

5 marks

7

Different alternative solutions are indicated with (Alt1), (Alt2), and so on.

(i) Many examples, including (for example) CCACCBCCCC, CCCACCBCCCCC, CCCCACCBCCCCCC. In fact, we can make any row $\mathsf{C}^x\mathsf{AC}^y\mathsf{BC}^z$ with $x+y=z$ and $y \geq 1$ and $y \leq x \leq 2y$. **2 marks**

(ii) Many examples violate these constraints, such as AB (every row contains at least one C), CBCACC (B to the left of A), CACBCCC (relation $x+y=z$ fails), CACCBCCC (relation $x \geq y$ fails), CCCACBCCCC (relation $x \leq 2y$ fails). **2 marks**

(iii) Yes. Begin with CACBCC and apply the join rule 5 times to the most recent row and itself, obtaining $\mathsf{C}^{32}\mathsf{AC}^{32}\mathsf{BC}^{64}$. Then begin again with the basic row CCACBCCC and similarly apply the join rule 4 times to reach $\mathsf{C}^{32}\mathsf{AC}^{16}\mathsf{BC}^{48}$. Combine these two results to achieve $\mathsf{C}^{64}\mathsf{AC}^{48}\mathsf{BC}^{112}$.

2 marks

(iv) (Alt1) Every achievable goal $\mathsf{C}^x\mathsf{AC}^y\mathsf{BC}^z$ satisfies $x \leq 2y$: the basic rows do so, and join moves preserve this property. So we cannot make this goal row.

(Alt2) The result of any game is a linear combination

$$\mathsf{C}^x\mathsf{AC}^y\mathsf{BC}^z = u \times \mathsf{CACBCC} + v \times \mathsf{CCACBCCC},$$

where $x = u + 2v$ and $y = u + v$. Putting $x = 128$ and $y = 48$ gives $u = -32$, $v = 80$, which is impossible. **3 marks**

(v) Writing $\mathsf{C}^x\mathsf{AC}^y\mathsf{BC}^z$ as $x+y=z$ and beginning with the basic rows $A : 1+1=2$ and $B : 2+1=3$, we can achieve $\mathsf{C}^{31}\mathsf{AC}^{16}\mathsf{BC}^{47}$ in 7 joins as follows

1. $3+2=5$ from A and B
2. $4+2=6$ from B and itself
3. $7+4=11$ from 1 and 2
4. $8+4=12$ from 2 and itself
5. $15+8=23$ from 3 and 4
6. $16+8=24$ from 4 and itself
7. $31+16=47$ from 5 and 6

(other sequences exist, including at least one with just six joins) **3 marks**

(vi) If a sequence doesn't join basic rows at each stage, then there's a slower sequence of joins that instead joins the rows that the non-basic row was made of.

A longest sequence begins with the basic row CACBCC and joins with the basic row CCACBCCC a total of 15 times to obtain the result $\mathsf{C}^{31}\mathsf{AC}^{16}\mathsf{BC}^{47}$.

We cannot do worse than this in a game where the final y value is 16. **3 marks**