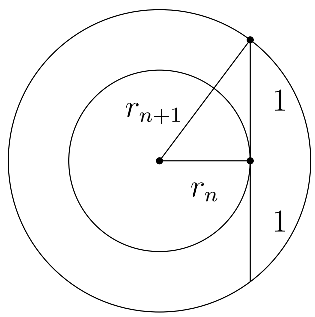
**SOLUTIONS FOR ADMISSIONS TEST IN**
MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS
**WEDNESDAY 02 NOVEMBER 2022**

**Mark Scheme:**
Each part of Question 1 is worth 4 marks which are awarded solely for the correct answer.
Each of Questions 2–7 is worth 15 marks

**1**

**A** Either $x \geq 0$ and $x^2 + 1 = 3x$ or $x < 0$ and $-x^2 + 1 = -3x$. The first equation has solutions $\frac{1}{2}\left(3 \pm \sqrt{5}\right)$ which are both positive. The second equation has solutions $\frac{1}{2}\left(3 \pm \sqrt{13}\right)$ and only one of these is actually negative. So there are three solutions in total.
**The answer is (d).**

**B** The relationship between circle $C_n$ of radius $r_n$ and circle $C_{n+1}$ of radius $r_{n+1}$ is shown below.




The tangent is at right-angles to the radius, so there's a right-angled triangle with hypotenuse $r_{n+1}$ and other sides $r_n$ and 1. Pythagoras gives $r_{n+1}^2 = r_n^2 + 1$. Since $r_1^2 = 1$, we have $r_2^2 = 2$ and $r_3^2 = 3$ and so on, up to $r_{100}^2 = 100$, so the radius of $C_{100}$ is 10.
**The answer is (d).**

**C** Complete the square for $x$ and for $y$ $(x - 2k)^2 - 4k^2 + (y - 2)^2 - 4 + 8 = k^3 - k$ This is the equation of a circle with centre $(2k, 2)$ and $r^2 = k^3 + 4k^2 - k - 4$ provided that expression is positive. Factorising the cubic as $(k + 4)(k - 1)(k + 1)$ reveals that this happens if and only if either $-4 < k < -1$ or if $k > 1$.
**The answer is (b).**

**D** $a_1 = 8 \times (3)^4$ and $a_2 = 8(8 \times 3^4)^4 = 8 \times 8^4 \times (3^4)^4$ and $a_3 = 8 \times \left(8 \times 8^4 \times (3^4)^4\right)^4$. In a similar way,

$$a_n = 8 \times 8^4 \times 8^{16} \times \cdots \times 8^{4^{n-1}} \times 3^{(4^n)}$$

The exponent on the 8s is $1+4+16+\cdots+4^{n-1} = \frac{1}{3}\left(4^n - 1\right)$. Then $8^{1/3} = 2$. Also use $4^n = 2^{2n}$.

$$a_n = 2^{2^{2n}-1} 3^{(2^{2n})} = \frac{6^{2^{2n}}}{2}$$

So for $n = 10$ we get $\frac{1}{2}6^{(2^{20})}$.
**The answer is (e).**

**E** We'll calculate the square of $(x + 1 + x^{-1})$ first;

$$x^2 + 2x + 3 + 2x^{-1} + x^{-2}$$

(I drew a square grid to keep track of all the cross-terms, and there's a nice pattern which helps). Now if we were to square this expression, the constant term independent of $x$ would be

$$2(x^2)(x^{-2}) + 2(2x)(2x^{-1}) + 3^2$$

Most of the terms have a factor of 2 because they occur in either order. This sum is $2+8+9 = 19$.
**The answer is (c).**

**F** Write $\alpha = \sin 72°$. We can substitute $\theta = 72°$ into the given equation, and then since $\sin(360°) = 0$, we get the equation

$$0 = 16\alpha^5 - 20\alpha^3 + 5\alpha$$

Either $\alpha = 0$ (which is not the case because 18° isn't an odd multiple of 90°), or we have the following quadratic equation for $\alpha^2$;

$$16(\alpha^2)^2 - 20(\alpha^2) + 5 = 0 \tag{1}$$

The quadratic formula followed by a square root, shows that $\alpha = \pm\sqrt{\frac{5\pm\sqrt{5}}{8}}$. One of these values is $\sin(72°)$, but what are the others? Thinking back to the point where we substituted in $\theta = 72°$, we can see that substituting $\theta = 36° \times n$ would have given the same equation for $\alpha$ for any whole number $n$, based on the positions of the zeroes of $\sin x$. So these roots are $\sin 36°$ and $\sin 72°$ and the negatives of those two expressions. We want $\sin 72°$ which is positive and larger than $\sin 36°$ so we should take the root with two $+$ signs.
**The answer is (a).**

**G** This expression is true for all real $n$, so write $n = m/\sqrt{2}$ and multiply through by 4 to get

$$m^4 + 4 = (m^2 + 2m + 2)(m^2 - 2m + 2)$$

Relabelling $m$ as $n$ gives us an interesting way to factorise $n^4 + 4$, so this expression is not prime unless at least one of $m^2 \pm 2m + 2 = 1$. Completing the square(s) gives $(m \pm 1)^2 + 1 = 1$ so $m = \pm 1$. We want positive whole numbers, so we just need to check that $1 + 4 = 5$ is prime to find the only example of a prime number of this form.
**The answer is (b).**

**H** We can use laws of logarithms to write the right-hand side of the given equation as

$$\log_2 \left(2x^3 + 6x^2 + 6x + 2\right).$$

Since $\log_2(x)$ is an increasing function for $x > 0$, we can compare the arguments of the logarithms, provided that both are positive. This gives the polynomial equation

$$2x^3 + 7x^2 + 2x + 3 = 2x^3 + 6x^2 + 6x + 2$$

which rearranges to $x^2 - 4x + 1 = 0$, which has two real solutions. We should check that $2x^3 + 7x^2 + 2x + 3$ is positive for these roots, but it definitely is because the roots of the quadratic are both positive and all the coefficients of the cubic are positive.
**The answer is (c).**

**I** The game is fair; the distribution of the number of heads that Alice gets is the same as the distribution of the number of tails that Bob gets. All we have to work out is the probability of a draw, and then subtract that from 1 and divide by 2.

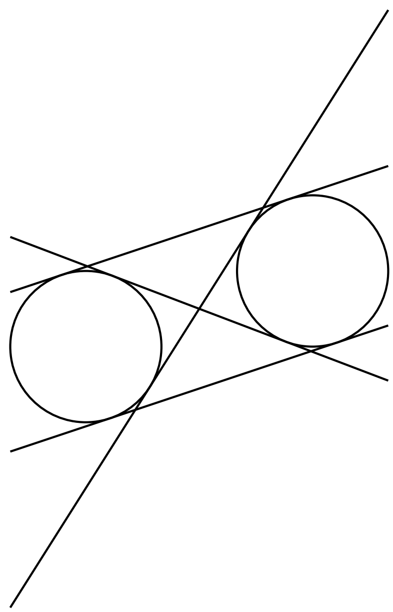
For a draw, both players would need to see the same number of their target side of the coin. There are

$$\binom{5}{0}^2 + \binom{5}{1}^2 + \binom{5}{2}^2 + \binom{5}{3}^2 + \binom{5}{4}^2 + \binom{5}{5}^2$$

ways that might happen, each with probability $\frac{1}{1024}$. Adding that up gives the probability of a draw to be $\frac{63}{256}$, so the probability that Alice wins is $\frac{1}{2} - \frac{63}{512} = \frac{193}{512}$.

Alternatively, count the ways that Alice wins (a bit slower).
**The answer is (a).**

**J** Write $y = mx + c$. A repeated root becomes a point $(x, y)$ where the line $y = mx + c$ is tangent to the curve $x^2 + y^2 = 1$ and also tangent to the curve $(x - 3)^2 + (y - 1)^2 = 1$. These are circles.



There are four lines that are tangent to both circles.
**The answer is (e).**

(i) $(x^2+Ny^2)^2-19(2xy)^2 = x^4+2Nx^2y^2+N^2y^4-76x^2y^2$ and we have $x^4-38x^2y^2+19^2y^4 = z^2$. So comparing coefficients we see that if $19^2 = N^2$ and $2N-76 = -38$ then the printed expression is equal to $z^2$. This happens if $N = 19$. **2 marks**

(ii) If $x = 13$ and $y = 3$ then $z = -2$.

Using the relationship in part (i), we get $(13 + 19 \times 3^2)^2 - 19(2 \times 13 \times 3) = (-2)^2$. So $340^2 - 19 \times 78^2 = 4$. **3 marks**

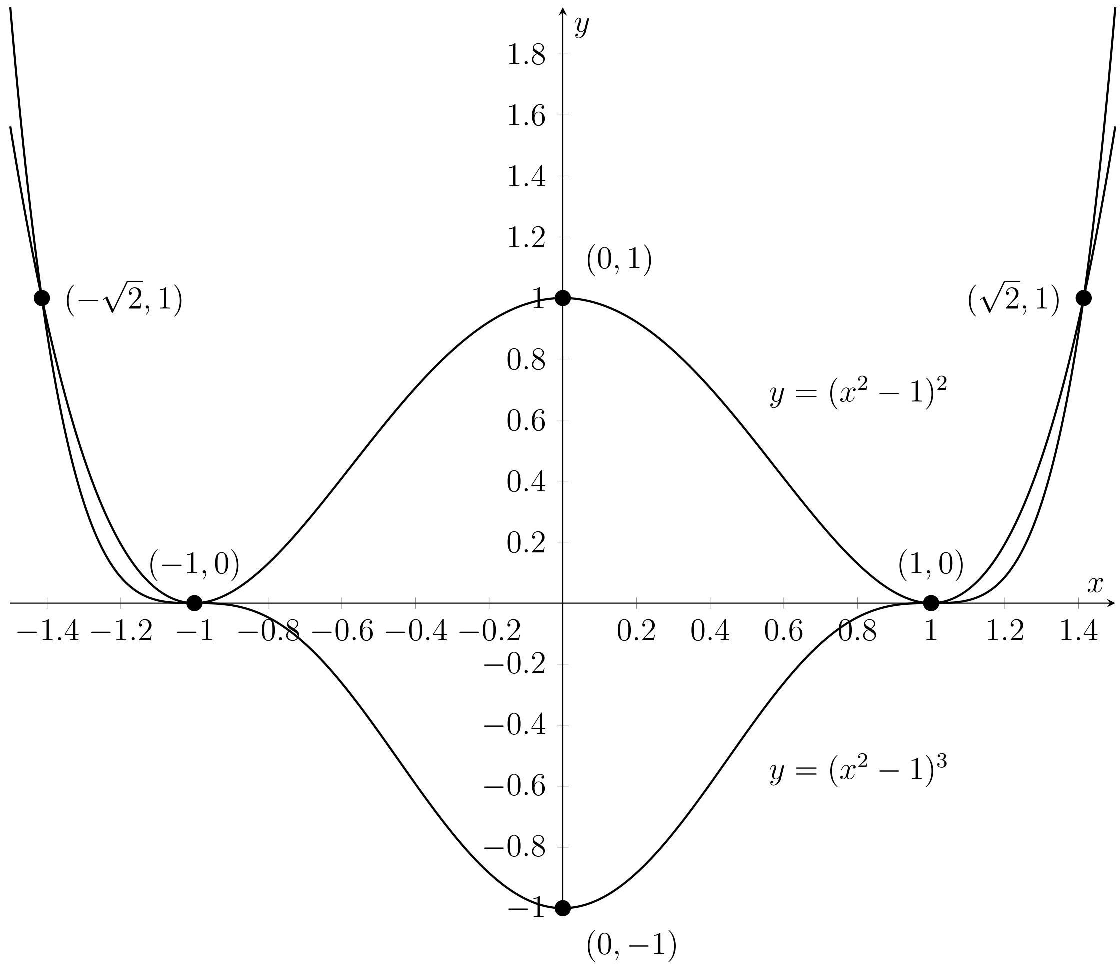
(iii) Now divide both sides by 4 to get $170^2 - 19 \times 39^2 = 1$. This is not the only solution; for example we could use the relationship in (i) to generate the solution $(170^2+19\times 39^2)^2-19(2\times 170\times 39)^2 = 1$. **3 marks**

(iv) Suppose $x^2 - 25y^2 = 1$. The left-hand side factorises to $(x-5y)(x+5y)$. Both of these brackets are whole numbers, so they must both be 1 or both be $-1$. But then adding be $2x = \pm 2$ and we want $x > 1$. So there are no solutions. **3 marks**

(v) Modify the relationship in part (i) to show that if $x^2-17y^2 = z$ then $(x^2+17y^2)-17(2xy)^2 = z^2$. Now note that $x = 4$ and $y = 1$ would give $4^2 - 17 \times 1 = -1$ so $z = -1$. Use these values of $x$ and $y$ in the relationship to generate the solution $(4^2 + 17 \times 1^2)^2 - 17(2 \times 4 \times 1) = 1$, or in other words $33^3 - 17 \times 8^2 = 1$. **4 marks**

**3**

(i) Sketch.




**3 marks**

(ii) $(x^2-1)^{2m} \geq 0$ so the integral can't be zero unless $a=0$. **1 mark**

(iii) Note first that $\int_0^1 (x^2-1)^{2m-1}\,\mathrm{d}x < 0$ and then $\int_1^a (x^2-1)^{2m-1}\,\mathrm{d}x$ is positive and increases without bound as $a$ increases. So at some point there is a value of $a_m$ such that the integral from 1 to $a$ balances out the negative contribution from the integral from 0 to 1. **2 marks**

(iv) $\int_0^{a_1} (x^2-1)\,\mathrm{d}x = 0$ so $\frac{a_1^3}{3} - a_1 = 0$. Now $a_1 > 0$ so $a_1 = \sqrt{3}$. **3 marks**

(v) Note that $\int_0^a (x^2-1)^3\,\mathrm{d}x = \frac{a^7}{7} - \frac{3a^5}{5} + a^3 - a$. Check that this is negative when $a=\sqrt{2}$ and positive when $a=\sqrt{3}$, so there's a change of sign and therefore a solution $a_2$ somewhere in between. To do the check, it's perhaps convenient to write the integral as

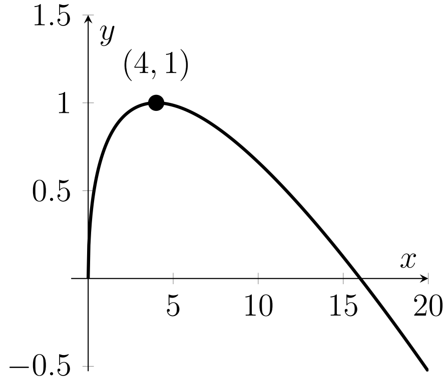
$$\frac{a}{35}\left(5a^6 - 21a^4 + 35a^2 - 35\right)$$

**3 marks**

(vi) Between 0 and $\sqrt{2}$ the integrand is very close to zero if $m$ is large; it's negative between 0 and 1 then positive from 1 to $\sqrt{2}$. We're told that the integral up to $\sqrt{2}$ is negative. But then for $x > \sqrt{2}$ the integrand quickly grows, and the integral will quickly cancel that negative value. So the value of $a_m$ will be close to $\sqrt{2}$ for large $m$. **3 marks**

**4**

(i) Sketch.




The turning point is at $(4, 1)$. **3 marks**

(ii) Note that $\sqrt{4x+1} - x - 1 = \sqrt{4x+1} - \frac{4x+1}{4} - \frac{3}{4}$. So this is a combination of a stretch and two translations. First squash by a factor of 4 parallel to the $x$-axis, then translate $\frac{1}{4}$ units to the left and $\frac{3}{4}$ units down.

The turning point is at $\left(\frac{3}{4}, \frac{1}{4}\right)$. **2 marks**

(iii) There are three cases depending on whether $x$ is to the left or to the right of each of $A$ and $B$. If $x < -1$ then we want $(1-x)(-1-x) = 1$ so $x = -\sqrt{2}$. If $-1 < x < 1$ then we want $(1-x)(1+x) = 1$ so $x = 0$. If $x > 1$ then we want $(x-1)(x+1) = 1$ so $x = \sqrt{2}$. So the three points are $(0, 0)$ and $(\pm\sqrt{2}, 0)$. **2 marks**

(iv) We have $((x-1)^2 + y^2)((x+1)^2 + y^2) = 1$. Expand and collect terms for a quadratic in $y^2$;

$$y^4 + 2(x^2+1)y^2 + x^4 - 2x^2 = 0$$

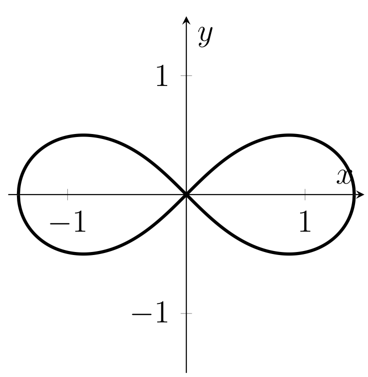
with (positive, real) solution

$$y = \sqrt{\sqrt{4x^2+1} - x^2 - 1}$$

**4 marks**

(v) This is the equation from (ii) with $x$ replaced by $x^2$ and with $y$ replaced with $y^2$. The turning point will now be at $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. **2 marks**

(vi) Sketch.




**2 marks**

**5**

(i) Pick $1, 3, 5, 7, 9, 11, 13$. **1 mark**

(ii) Pick $1, 3, 5, \ldots, 2k+1$. The items in odd positions are untouched when first picked. For example, when door $2l+1$ is first picked, no door numbered greater than $2l-1$ has been picked before that, so the item behind door $2l+1$ is what it originally was. Thus each opened door results in a new item for Alice and $k+1$ doors are opened. **2 marks**

(iii) Pick $1, 1, 4, 4, 7, 7, 10, 10, 13, 13$. Any permutation of this sequence works. Other sequences may work. **1 mark**

(iv) Pick $1, 1, 4, 4, \ldots, 3l+1, 3l+1, \ldots, 3k+1, 3k+1$. (Permutations allowed.) Group the items as,

$$(a_1, a_2), (a_3, a_4, a_5), (a_6, a_7, a_8), \ldots, (a_{3k}, a_{3k+1}).$$

Observe that the first time any door from each group is picked, the items are in their original position. Except at the boundaries, the door that is picked is always the one in the middle, i.e. a door of the form $3l+1$. Thus Alice wins at least 2 items from each group, and there are $k+1$ groups. **2 marks**

(v) (a) Pick $1, 1, 3$. Many permutations work: $(1, 3, 1)$, $(3, 3, 1)$, $(3, 1, 3)$. But $(1, 3, 3)$ or $(3, 1, 1)$ don't work. Anything involving 2 also doesn't work. **1 mark**

(b) Pick $1, 1, 5, 5, 3$. Note that 1 and 5 have to be picked at least once before picking 3; any ordering of $(1, 1, 5, 5, 3)$ that respects this property will work. No other sequence works; this can be checked by (somewhat painful) exhaustive enumeration, reducing cases by branch and bound. **1 mark**

(vi) Pick $1, 1, 5, 5, 9, 9, 13, 13, 3, 7, 11$. Note that 1 and 5 have to be picked at least once each before picking 3; 5 and 9 have to be picked at least once each before picking 7, and so on. Any permutation that respects these conditions works just fine as the proof below shows. **1 mark**

(vii) Pick $1, 1, 5, 5, \ldots, 4l+1, 4l+1, \ldots, 4k+1, 4k+1$, then followed by $3, 7, \ldots, 4l+3, \ldots, 4k-1$. This is a total of $3k+2$ door choices. Group items as below.

$$(a_1, a_2), (a_3), (a_4, a_5, a_6), (a_7), \ldots, (a_{4l-1}), (a_{4l}, a_{4l+1}, a_{4l+2}) \ldots, (a_{4k-1}), (a_{4k}, a_{4k+1}).$$

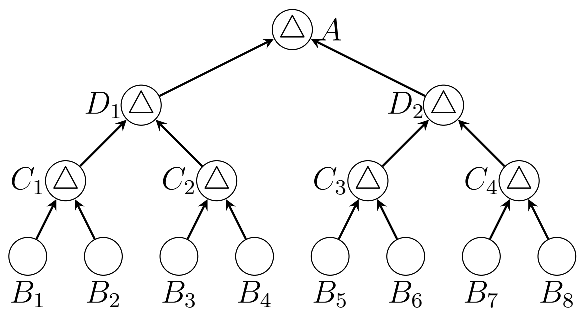
Note that Alice is guaranteed to get at least two items from the groups that have 2 or 3 elements each and picks up all the items behind doors numbered $4l+3$ for some $l$. **2 marks**

(viii) No. Since we only need to show that Alice can't *guarantee* winning all items, it is OK to assume that the host knows Alice's entire sequence (which may be longer than 6). The host can always deny Alice one of items $a_3$ or $a_4$. Whenever Alice picks either door 1 or 2, the host will always swap items behind doors 1 and 2; likewise for doors 5 and 6. Now suppose for the first time Alice picks door 3 or 4. Without loss of generality this is door 3; then the host can permanently deny item $a_4$ to Alice. Depending on whether the next pick among $\{3, 4\}$ is 3 or 4, the host can always guarantee that item $a_4$ is behind the door that Alice won't pick next. For noticing the importance of 3 and 4; for proper justification **4 marks**

(i) No. Suppose $A$, $B$, $C$, don't have the same opinion, e.g. $A$ is $\triangle$ and the other two are $\square$. Then after one round $B$ will have $\triangle$, but $A$ and $C$ will have $\square$; then after the second round $C$ will have $\triangle$, but $A$ and $B$ will have $\square$; and after the third round we are back to the starting configuration and this repeats indefinitely. **2 marks**

(ii) Yes. Label the layers, $1, 2, 3, 4$ from the bottom. Nodes in layer 1 never change their opinion. Nodes in layer 2 may change their opinion after the first round but never subsequently, and so on. So after three rounds, no opinions will change in the network. **3 marks**

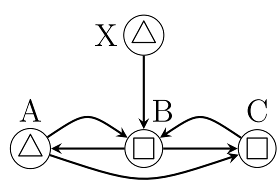
(iii) The network below works.




As argued in the previous part, we have 4 layers, and the layers remain static after rounds 0, 1, 2, 3 respectively from the bottom. The node $C_1$ changes its opinion only if $B_1$ and $B_2$ are both $\square$, the same for $C_2$, wrt $B_3$ and $B_4$, etc. So if not all $B_i$ are $\square$, at least one of the $C_i$'s remains $\triangle$ forever. Likewise, if one of the $C_i$'s is $\triangle$ forever, then one of the $D_i$'s remains $\triangle$ forever. In which case $A$ would retain its $\triangle$ opinion. However, if all $B_i$ were $\square$, then after one round all $C_i$ would become $\square$; after two rounds all $D_i$ would become $\square$ and then finally after three rounds $A$ would become $\square$ and the network would remain stable. **3 marks**

(iv) No. Consider the combined network below.

$N_1 + N_2$




Note that after one round, $X$ and $C$ will not have changed their opinion, but $A$ and $B$ will both have flipped their opinions. After the second round, we return to the starting position. This continues indefinitely. **3 marks**

(v) (a) $2^n$. **1 mark**

(b) Simulate the influencer network dynamics for $2^n+1$ rounds. Since there are only $2^n$ possible configurations, at least two assignments are equal. If the network is stable, the period of the system must be 1, so the assignment at time $2^n$ and $2^n+1$ must be the same. Otherwise, the system is not stable. **3 marks**

(i) Not all are achievable. We cannot have both the odd tokens in decreasing order and the even tokens in increasing order. So the valid sequences (531642) and (642531) are not achievable.
**2 marks**

(ii) There are 8. All the even tokens must be together and likewise all the odd ones. We have either evens first or second (2 choices), the evens could be increasing or decreasing (2 choices), the odds could be increasing or decreasing (2 choices), multiplying together gives $2^3 = 8$ valid sequences. **3 marks**

(iii) The data operator can achieve 6 sequences. All sequences where the odd tokens are increasing are possible. When the odds are first, we just *pass* all odd tokens to output; depending on whether the evens are increasing or decreasing, we either first use pushL or pushR on all evens, and then eventually pop them all. When the odds are second, we always use pushL on the odds at the start, for evens if they are increasing use pass directly, otherwise, use pushR on them. Eventually all (remaining) tokens are in storage in the right order and then just pop everything.

When odds are decreasing, they must be all pushed in using pushR, if the evens are increasing and appear first, they are directly sent to output using pass, if they appear second, then they are pushed in using pushL and eventually everything is output using pop.

Making both odds and evens in decreasing order is not possible, as in order to make sequences reverse the input order they must be stored using pushR; however, in this case they must be interleaved in the storage unit, which makes the sequence invalid. **4 marks**

(iv) There are a total of $3! \cdot 2^3 = 48$ such sequences. Call the three groups of tokens $G_0, G_1, G_2$, where $G_i$ is all tokens that are of the form $3k + i$. The groups can appear in any order, giving 3! choices, and then within each group we have two choices for each (increasing or decreasing).
**3 marks**

(v) There are only $3! = 6$ possible achievable sequences. Note that the three groups defined above $G_0, G_1, G_2$ can't be mixed in the storage unit. So one of the group needs to go straight to the output channel using pass operations. This will appear as the first group in the output channel and has to be in increasing order. For the other two groups, one will all have to be stored using pushL operations and the other by pushR operations to avoid mixing. Thus, the second group that is output will have to be in decreasing order and the last group in increasing order again. So we no longer have any choice on increasing/decreasing order within the groups, but we can decide which group goes first (use pass directly on them), which goes second (use pushR and then pop after the first group is finished), and which goes last (use pushL and then pop after the first two groups are finished). **3 marks**