SOLUTIONS FOR ADMISSIONS TEST IN
MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS
OCTOBER 2023

Mark Scheme:
Each part of Question 1 is worth 4 marks which are awarded solely for the correct answer.
Each of Questions 2–7 is worth 15 marks

1

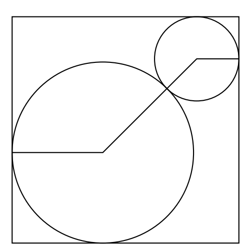
A Each of these can be written as $M + \log_{10} N$ with M an integer and N a number between 9 and 10. That's close to $M + 1$, and we should pick the one with N closest to 10.

$$2\beta = \log_{10} 9, \qquad 5\alpha + \beta = 1 + \log_{10} 9.6, \qquad \alpha + 2\gamma = 1 + \log_{10} 9.8,$$
$$2\alpha + 5\beta = 2 + \log_{10} 9.72, \qquad 2\alpha + \beta + \gamma = 1 + \log_{10} 8.4$$

The answer is (c).

B Not (a), that's one less than a square. Not (b), that ends in 3 (the last digit of a square just depends on the last digit of the number being squared, and checking 0–9 reveals that 3 is impossible for the last digit of the square). Not (d), that's a multiple of 5 but not 25 (last two digits). Not (e), that's a multiple of 1000 but not 10000 (count the zeroes). So (c) must be the square.
The answer is (c).

C Call the radii r_1 and r_2. Then drawing lines from the centres to the points of tangency and doing some trigonometry reveals that $r_1 + (r_1 + r_2)/\sqrt{2} + r_2 = 1$.



So we can deduce the value of $r_1 + r_2 = \frac{1}{1+1/\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}+1} = 2 - \sqrt{2}$ without actually solving for r_1 and r_2 separately.
The answer is (b).

D Work from the outside inwards

$$\begin{aligned}
((x^2-1)^2-2)^2-3 &= 2 \quad\text{or}\quad -2\\
((x^2-1)^2-2)^2 &= 5 \quad\text{or}\quad 1\\
(x^2-1)^2-2 &= \sqrt{5} \quad\text{or}\quad -\sqrt{5} \quad\text{or}\quad -1 \quad\text{or}\quad 1\\
(x^2-1)^2 &= 2+\sqrt{5} \quad\text{or}\quad 2-\sqrt{5} \quad\text{or}\quad 1 \quad\text{or}\quad 3\\
x^2-1 &= \sqrt{2+\sqrt{5}} \quad\text{or}\quad -\sqrt{2+\sqrt{5}} \quad\text{or}\quad 1 \quad\text{or}\quad -1 \quad\text{or}\quad \sqrt{3} \quad\text{or}\quad -\sqrt{3}\\
x^2 &= 1+\sqrt{2+\sqrt{5}} \quad\text{or}\quad 1-\sqrt{2+\sqrt{5}} \quad\text{or}\quad 2 \quad\text{or}\quad 0 \quad\text{or}\quad 1+\sqrt{3} \quad\text{or}\quad 1-\sqrt{3}
\end{aligned}$$

Where we have used the fact that $2-\sqrt{5}<0$. Now note that $\sqrt{2+\sqrt{5}}>\sqrt{2}>1$ so $1-\sqrt{2+\sqrt{5}}$ is (very) negative. Also, $1-\sqrt{3}$ is negative. The number of real solutions is therefore seven; two from each positive value of x^2 and one from the zero solution.
The answer is (c).

E The sum of all the numbers up to 3^{10} is an arithmetic series. The sum of all the powers of 3 up to 3^{10} is a geometric series. We want the difference.

$$\begin{aligned}&\left(1+2+3+4+\cdots+(3^{10}-1)+3^{10}\right)-\left(1+3+3^2+\cdots+3^{10}\right)\\&=3^{10}\frac{3^{10}+1}{2}-\frac{3^{11}-1}{2}=\frac{(3^{10})^2-2\times 3^{10}+1}{2}=\frac{(3^{10}-1)^2}{2}\end{aligned}$$

The answer is (a).

F Multiply out $(1-2x)(1+2x)$ first to get $(1-4x^2)$, then multiply that by $(1+4x^2)$ to get $(1-16x^4)$, then look for the $\binom{5}{3}$ coefficient to get $\binom{5}{3}(-16x^4)^3=-10\times 2^{12}x^{12}$.
The answer is (a).

G Given $b^2=4ac$ and $c^2=4ab$, we can multiply for $b^2c^2=16a^2bc$. Since $bc\neq 0$ we have $a^2=\frac{1}{16}bc$. The discriminant of the third quadratic is $a^2-4bc=-63a^2<0$. So this has no real roots.

Alternatively, solve $b^2=4ac$ and $c^2=4ab$ in terms of a (multiply the first by b and the second by c to see that $b^3=c^3$) for $b=4a$ and $c=4a$. Then $a^2-4bc=-63a^2$ is negative.
The answer is (a).

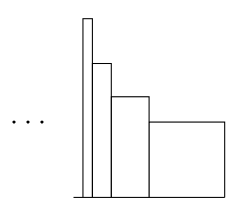
H Consider the area formula $\frac{1}{2}ab\sin\gamma$ with $a=b=10$ fixed. We would maximise this by making $\sin\gamma$ large, which occurs when γ is close to $90°$, which is when the opposite side is close to $\sqrt{2}$ times the length of the short sides. Note that $\sqrt{2}\approx 3/2$ because $2\approx 9/4$, so the triangle with the largest area is the $(10,10,15)$ triangle.
The answer is (d).

I $p(x)=Ax(x-M)^2$ because 0 is a root and M is a repeated root. Now set $p(1)=1$ and $p(2)=2$ for

$$1=A(1-M)^2 \quad\text{and}\quad 2=2A(2-M)^2$$

Eliminate A for the quadratic $(2-M)^2=(1-M)^2$. Solve; either $2-M=1-M$ which is nonsense, or $2-M=M-1$, which gives solution $M=3/2$.
The answer is (e).

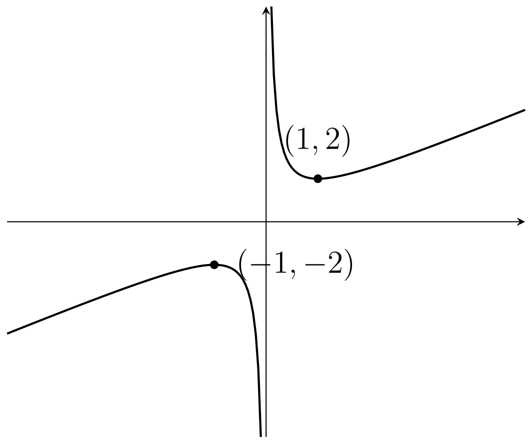
J The area under the graph is made up of infinitely many rectangles, each half the width and 4/3 the height of the previous, reading from right-to-left.



The areas of the rectangles form a geometric progression, so the total area is a geometric series with first term 1 and common ratio 2/3.
The answer is (c).

2

(i) Sketch $p_1(x) = x + 1/x$.



Turning points are at $1 - x^{-2} = 0$ so $x = \pm 1$. Corresponding values of y are ± 2 respectively.

3 marks

(ii) We have $p_1(x)^2 - 2 = (x + x^{-1})^2 - 2 = x^2 + 2 + x^{-2} - 2 = p_2(x)$ **1 mark**

(iii) We have $p_1(x)^3 = x^3 + 3x + 3x^{-1} + x^{-3} = p_3(x) + 3p_1(x)$ so $p_3(x) = p_1(x)^3 - 3p_1(x)$.

1 mark

(iv) x isn't zero, so divide by x^2. Then the left-hand side becomes $p_2(x) + p_1(x) - 10$ which is $p_1(x)^2 + p_1(x) - 12$. That's a quadratic for $p_1(x)$ which factorises as $(p_1(x) - 3)(p_1(x) + 4) = 0$, so $p_1(x) = 3$ or $p_1(x) = -4$. Each of those is a quadratic for x; we have

$$x^2 - 3x + 1 = 0 \qquad \text{or} \qquad x^2 + 4x + 1 = 0$$

The solutions are $x = \frac{3 \pm \sqrt{5}}{2}$ or $x = -2 \pm \sqrt{3}$. **5 marks**

(v) First note that $x = 1$ is a solution, and divide by $(x - 1)$.

Now adapt the work in the previous part; divide by x^3 to get $p_3(x) + 3p_2(x) - 2p_1(x) - 9 = 0$. Now write this in terms of $p_1(x)$ using part (ii) and factorise;

$$\begin{aligned} p_3(x) + 3p_2(x) - 2p_1(x) - 9 &= p_1(x)^3 - 3p_1(x) + 3p_1(x)^2 - 6 - 2p_1(x) - 9 \\ &= p_1(x)^3 + 3p_1(x)^2 - 5p_1(x) - 15 \\ &= (p_1(x) + 3)(p_1(x)^2 - 5) \\ &= (p_1(x) + 3)(p_1(x) - \sqrt{5})(p_1(x) + \sqrt{5}) \end{aligned}$$

Solve $p_1(x) = -3$ for $x = \frac{-3 \pm \sqrt{5}}{2}$, solve $p_1(x) = \sqrt{5}$ for $\frac{\sqrt{5} \pm 1}{2}$, and solve $p_1(x) = -\sqrt{5}$ for $\frac{-\sqrt{5} \pm 1}{2}$.

5 marks

3

(i) The argument is almost irrelevant; it takes all real values, so the maximum of the cosine is 1. **1 mark**

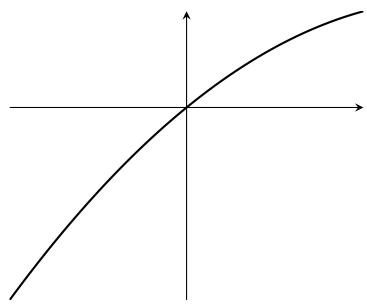
(ii) Write $u = \cos(2x + 30°)$ and this becomes a quadratic $u(1 - u)$ in u. The maximum of a quadratic occurs midway between the roots, in this case at $u = 1/2$, where the value is $1/4$. Note that $u = 1/2$ is a value which can actually be obtained by $u = \cos(2x + 30°)$ **4 marks**

(iii) This is the previous function raised to the power of 5, so the maximum value occurs at the same value of u where $u(1 - u)$ was maximised in the previous part of the question. The value is now $4^{-5} = 2^{-10} = \frac{1}{1024}$. **4 marks**

(iv) First use $\cos(90° - x) = \sin(x)$ and $\cos^2 x + \sin^2 x = 1$ to write this as

$$(\sin(3x - 60°))^8 \, (3 - \sin(3x - 60°))^8$$

Now write $u = \sin(3x - 60°)$ to make this $u^8(3 - u)^8$. Ignoring the power of 8 for a moment, the function $u(3 - u)$ looks like this for $-1 \leq u \leq 1$;



That is, the function takes extreme values at the end-points (because the turning point is out of range at $u = 1.5$). We should note that at $u = -1$, the value of this quadratic is -4, while at $u = 1$, the value is 2.

This function is raised to the power of 8, and it's the value of $u = -4$ which gives the maximum. The value is $4^8 = 2^{16}$. **6 marks**

(i) The normal at $x = a$ is

$$y = -\frac{1}{a}(x-a) + \frac{a^2}{2}.$$

Let B have coordinates (x_b, y_b). Then the distance from A is d so

$$(x_b - a)^2 + \left(y_b - \frac{a^2}{2}\right)^2 = d^2$$

Substitute in the equation of the line and simplify for $(x_b - a)^2(1 + \frac{1}{a^2}) = d^2$.

B is above and to the left of A and $a > 0$ so $x_b < a$.

The coordinates of B are therefore

$$x_b = a - \frac{d}{\sqrt{1 + 1/a^2}} = a - \frac{ad}{\sqrt{1+a^2}} \quad \text{and} \quad y_b = \frac{a^2}{2} + \frac{d}{\sqrt{1+a^2}}$$

6 marks

(ii) We would need $y_b = \frac{1}{2}x_b^2$ so

$$\frac{a^2}{2} + \frac{L}{\sqrt{1+a^2}} = \frac{1}{2}\left(a - \frac{ad}{\sqrt{1+a^2}}\right)^2$$

Expand and simplify this to get

$$\frac{d}{\sqrt{1+a^2}} + \frac{a^2 d}{\sqrt{1+a^2}} = \frac{a^2 d^2}{2(1+a^2)}$$

Divide by d and multiply by $2\sqrt{1+a^2}$.

$2(1+a^2)^{3/2} = a^2 d$.

4 marks

(iii) Divide by a^2 and raise both sides to the power of 2/3.

$d^{2/3} = 2^{2/3}(1+a^2)/a^{4/3} = f(a^2)$ where $f(t) = 2^{2/3}(1+t)/t^{2/3} = 2^{2/3}(t^{-2/3} + t^{1/3})$. **2 marks**

(iv) $t^{-2/3} + t^{1/3}$ is minimised where $-\frac{2}{3}t^{-5/3} + \frac{1}{3}t^{-2/3} = 0$ i.e. $t = 2$. The value of $f(t)$ there is 3. So if $d < 3\sqrt{3}$ then d is lower than the minimum and there's no such value of a. **3 marks**

5

(i) $F_3 = 2, F_4 = 3, F_5 = 5$. **3 marks**

(ii) It takes $n-2$ additions. One each to compute $F_3, F_4, \dots, F_n$. (Any candidate giving any of $n-2, n-1, n$ as answers should not be penalised.) **1 mark**

(iii) $S_1 = 2$, as (0) and (1) are both valid. $S_2 = 3$ as $(0,0), (0,1)$, and $(1,0)$ are valid, but $(1,1)$ is not. (Explanations not required.) **1 mark**

(iv) Suppose the first element is 0, then any valid sequence of length $n-1$ could be appended to it to get a valid sequence of length n. (1 mark) If the first element is 1, the subsequent element has to be 0, after that any valid sequence of length $n-2$ could be appended, so we get $S_n = S_{n-1} + S_{n-2}$. (1 mark) We observe that $S_1 = F_3$ and $S_2 = F_4$; as F_n and S_n satisfy the same equation (*), $S_n = F_{n+2}$ for all n. (1 mark) **3 marks**

(v) Consider a valid sequence of length $2n-3$ as suggested. If the element at position $n-1$ is 0, then adding any possible valid sequence of length $n-2$ as a prefix before it and another such sequence as a suffix after it, gives a valid sequence of length $2n-3$. (1 mark) If the element is 1, then the elements in position $n-2$ and n must be 0, but the sequences from positions 1 to $n-3$ and $n+1$ to $2n-3$ could each be an arbitrary valid sequence of length $n-3$. (1 mark). So we get,

$$S_{2n-3} = S_{n-2}^2 + S_{n-3}^2.$$

Since $S_n = F_{n+2}$ that completes the proof. **2 marks**

(vi) Consider a valid sequence of length $2n-2$, and focus on the element at position $n-1$. (1 mark) If it is 0, then the sequence from 1 to $n-2$ could be any valid sequence of length $n-2$; likewise the sequence from n to $2n-2$ could be any valid sequence of length $n-1$. If it is 1, then the elements at position $n-2$ and n must be 1; the sequence from 1 to $n-3$ could be any valid sequence of length $n-3$ and the one from $n+1$ to $2n-2$ could be any valid sequence of length $n-2$. (1 mark) So we get,

$$S_{2n-2} = S_{n-2}S_{n-1} + S_{n-3}S_{n-2}.$$

Using $S_n = F_{n+2}$,

$$\begin{aligned} F_{2n} &= F_n F_{n+1} + F_{n-1}F_n \\ &= F_n(F_n + F_{n-1}) + F_{n-1}F_n \\ &= F_n^2 + 2F_{n-1}F_n. \end{aligned}$$

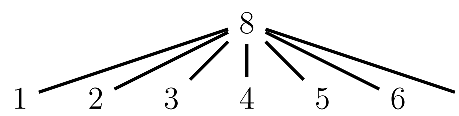
(1 mark) **3 marks**

(vii) We should compute F_{2^i} and F_{2^i-1} for $i = 2, \dots, k-1$ (F_2 and F_1 are known), and then use $F_{2^{k-1}}$ and $F_{2^{k-1}-1}$ to compute F_{2^k}. Each pair requires 3 multiplications (two squares and one product) and two additions. So 5 operations each for $i = 2, \dots k-1$ and 3 operations for F_{2^k}, giving $5(k-2)+3 = 5k-7$. (Total 2 marks, at least 1 mark should be given for any reasonable answer that is $O(k)$ and 2 marks given for anything remotely sensible, such as $3k, 4k, 5k, 6k$ etc.) **2 marks**

6

(i) The code for T_2 is **6165886**. **1 mark**

(ii) The following tree has code **8888888**.



1 mark

(iii) Since **8** doesn't appear in the code, it must be a leaf; (1 mark) no other node is a leaf, so in fact the tree is a line. For ease of display, we'll show the tree horizontally rather than vertically. (1 mark for correct tree)

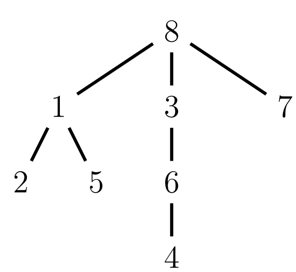
$$8 — 3 — 1 — 6 — 5 — 4 — 7 — 2$$

2 marks

(iv) The leaves are $2, 4, 5$ and 7. The leaves must be the digits that don't appear in the code as they are not anyone's parent. (2 marks for correct answer; 1 mark for justification) **3 marks**

(v) The first digit that was deleted has to be the smallest leaf, which in this case is 2, since **1** is the first digit in the sequence, 1 must be the parent of 2. Since 1 appears subsequently in the sequence, even after deletion of 2 it can't be a leaf. The next leaf deleted must be 4 which has 6 as its parent. 6 now becomes a leaf; but the next smallest leaf to be deleted is 5 for which 1 must be the parent, at which point 1 becomes a leaf, as it no longer appears in the rest of the sequence. So the two children of 1 are 2 and 5. (1 mark for each correctly identified child; no need for justification as the question doesn't ask for it and the justification will be awarded marks in future parts.) **2 marks**

(vi) (2 marks) This is the tree:



2 marks

(vii) We identify the smallest leaf and the first digit in the code must be its parent. Subsequently in turn, we determine if any new leaf has been created (by deletion) and then again identify the smallest remaining leaf and determine its parent. In this manner we have obtained the parent of every digit (except the last digit in the code which must be the root). Once the parent of each non-root node is known, we can easily reconstruct the tree. **2 marks**

(viii) Every tree has a unique code. (1 mark) There are $8^7 = 2^{21} = 2 \cdot (2^{10})^2$ possible codes which is greater than $2 \cdot 10^6$. (1 mark) **2 marks**