Weak solutions to the isentropic system of gas dynamics

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Outline

1. Weak solutions to the Incompressible flows
    - Onsager's conjecture
    - ∞ many admissible solutions

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1. Weak solutions to the Incompressible flows
   - Onsager's conjecture
   - ∞ many admissible solutions

2. Construction of ∞ many global admissible weak solutions
   - Main result
   - Key idea and steps

Incompressible Euler equations

- A solution $(\mathbf{u}, P)$ to the incompressible Euler equations is such that

$$\begin{cases} \partial_t \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla P = 0, \quad x \in \mathbb{T}^3, \\ \operatorname{div} \mathbf{u} = 0. \end{cases}$$

If the solution is sufficiently smooth, say C^1, then the total *kinetic energy*

$$E(\mathbf{u}) := \frac{1}{2} \int_{\mathbb{T}^3} |\mathbf{u}(t,x)|^2 \, dx$$

is conserved, and any solution is uniquely determined by the initial data.

- A folklore conjecture: Uniqueness should fail when $\mathbf{u} \in C^\alpha$ for some $\alpha < 1$, which is highly linked to Onsager's conjecture.
- Question: Can we construct $+\infty$ many global admissible weak solutions?
    - It will narrow down further the class of weak solutions to single out physical relevant solutions of the Euler equations for the uniqueness.

Onsager's semi-formal proof of the sufficient condition

- Roughly speaking, enough regularity allows us to control convective term and to do integration by parts.

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- Roughly speaking, enough regularity allows us to control convective term and to do integration by parts.

- The term to control is the total energy flux

  $$\Pi = \langle \mathrm{div}(\mathbf{u} \otimes \mathbf{u}), \mathbf{u} \rangle \sim \left\langle (\nabla^{1/3}\mathbf{u} \otimes \nabla^{1/3}\mathbf{u}) : \nabla^{1/3}\mathbf{u} \right\rangle$$

  Thus the quantity $\|\nabla^{1/3}\mathbf{u}\|_{L^3}$ appears. Any better regularity would be sufficient to justify integration by parts to show that the flux $\Pi = 0$.

Onsager's Conjecture [Onsager '49]

The threshold Hölder regularity for the validity of the energy conservation of weak solutions has exponent 1/3:

(1) Every weak solution $\mathbf{u}$ to the Euler equations with Hölder continuity exponent $\alpha > \frac{1}{3}$ conserves energy.

(2) For any $\alpha < \frac{1}{3}$ there exists a weak solution $\mathbf{u} \in C^\alpha$ which does not conserve energy.

Threshold regularity

energy not conserved

Scheffer '93
Shnirelman '97
De Lellis-Székelyhidi '12
B-DL-I-S '13
I '16, B-DL-S-V '17

energy conserved

Eyink '94
Constantin-E-Titi '94
Duchon-Robert '00
......

Onsager

$C^{1/3}$

non-uniqueness

C^1

uniqueness

De Lellis-Székelyhidi '09, '10
Daneri '14
Daneri-Székelyhidi '17
Daneri-Runa-Székelyhidi '20

based on a Baire category argument

Weak solutions to the Cauchy problem

$$\begin{cases} \partial_t \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, & x \in \Omega, \\ \operatorname{div} \mathbf{u} = 0, \\ \mathbf{u}|_{t=0} = \mathbf{u}^0. \end{cases}$$

A divergence free vector field $\mathbf{u} \in L_t^\infty L_x^2$ is a *global admissible weak* solution if

- $\int_0^\infty \int_\Omega \left(\mathbf{u} \cdot \partial_t \varphi + \mathbf{u} \otimes \mathbf{u} : \nabla \varphi\right) dx dt = -\int_\Omega \mathbf{u}^0 \cdot \varphi(\cdot, 0) dx$
  for every test function $\varphi \in C_c^\infty$ with $\operatorname{div} \varphi = 0$.
- $\int_\Omega \frac{1}{2} |\mathbf{u}(\cdot, t)|^2 dx \leq \int_\Omega \frac{1}{2} |\mathbf{u}^0(\cdot)|^2 dx \quad$ for every $\; t \geq 0$.

Non-uniqueness and density of 'wild' data

Theorem (Székelyhidi-Wiedemann '12, Chen-Vasseur-Y.)

For any $\varepsilon > 0$ and any $\mathbf{u}^0 \in L^2(\mathbb{T}^n)$, there exist infinitely many $v^0 \in L^2(\mathbb{T}^n)$ satisfying

$$\|v^0 - \mathbf{u}^0\|^2_{L^2(\mathbb{T}^n)} < \varepsilon,$$

such that for each such initial value v^0, there exist infinitely many global admissible weak solutions v to the incompressible Euler equations.

- Construct a sub-solution by vanishing viscosity limit from Navier-Stokes.
  - Leray-Hopf theory for N.-S.
  - Euler equations: No results of global existence of weak solutions.
  - Inviscid limit ($\mu \to 0$): weak limit is not commutative with nonlinear term.
- Applying C.I. to sub-solution to generate ∞ many weak solutions.

Isentropic Euler system

Weak solutions

$$\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0,$$
$$(\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P = 0$$

- $\int_0^\infty \int_\Omega \left(\rho \partial_t \varphi + V \cdot \nabla \varphi\right) dxdt = -\int_\Omega \rho^0 \varphi(\cdot, 0) dx$
- $$\int_0^\infty \int_\Omega \left(V \cdot \partial_t \varphi + \frac{V \otimes V}{\rho} : \nabla \varphi + \rho^\gamma \,\mathrm{div}\, \varphi\right) dxdt$$
  $$= -\int_\Omega V^0 \cdot \varphi(\cdot, 0) dx$$
  where $V = \rho \mathbf{u}$.
- $\int_\Omega \left(\frac{|V|^2}{2\rho} + \frac{\rho^\gamma}{\gamma - 1}\right) dx \leq \int_\Omega \left(\frac{|V^0|^2}{2\rho^0} + \frac{(\rho^0)^\gamma}{\gamma - 1}\right) dx.$

Related works

- The proof relies on the *Convex integration* machinery developed by De Lellis–Székelyhidi.
- Two directions of the isentropic flow
  - One direction, pioneered by Chiodaroli, considers a wide class of initial densities. Some extensions, Luo–Xie–Xin, and Feireisl.
  - The other direction, pioneered by Chiodaroli–De Lellis–Kreml, focuses on initial values being Riemann data.
  - Extensions of both strategies have been studied for the full Euler system, see Chiodaroli–Feireisl–Kreml, Al Baba–Klingenberg–Kreml–Mácha–Markfelder.
- Without energy condition, non-unique solutions can be constructed for any fixed initial values, see Abbatiello–Feireisl.
- A natural problem consists in studying the size of the class of initial values leading to non-unique solutions.

Riemann data

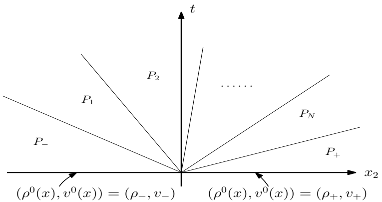
Theorem (*Chiodaroli- De Lellis-Kreml,CPAM.)*

For $\gamma = 2$ in 2D, there are infinitely many bounded admissible solutions with the initial data

$$(\rho^0, \boldsymbol{u}^0) = \begin{cases} (\rho_-, \boldsymbol{u}_-), & \text{if } x_2 < 0 \\ (\rho_+, \boldsymbol{u}_+), & \text{if } x_2 > 0. \end{cases}$$

- Admissible condition: energy inequality in distribution sense.
- Initial data is Riemann data.
- Key idea: sub-solutions+ convex integral.

Key idea of CDK



- Classical theory in 1D conservation laws: Rankine-Hugoniot conditions.
- Sub-solutions: $(\bar{\rho}, \bar{\mathbf{u}}) = \sum_{-}^{+} (\rho, \mathbf{u}) \mathbb{1}_{P_i}$
- Oscillation lemma: Let $\bar{\mathbf{u}} \otimes \bar{\mathbf{u}} - \bar{U} < \frac{C}{n} Id$, there exists infinitely many bounded maps $(\underline{\mathbf{u}}, \underline{U}) \in L^\infty$, such that
  - $\underline{\mathbf{u}}, \underline{U}$ vanish identically outside Ω,
  - $\operatorname{div} \underline{\mathbf{u}} = 0, \ \underline{\mathbf{u}}_t + \operatorname{div} \underline{U} = 0;$
  - $(\bar{\mathbf{u}} + \underline{\mathbf{u}}) \otimes (\bar{\mathbf{u}} + \underline{\mathbf{u}}) - (\bar{U} + \underline{U}) = \frac{c}{n} Id.$
- Solutions: $(\rho, \mathbf{u}) = (\bar{\rho}, \bar{\mathbf{u}} + \underline{\mathbf{u}})$.

Our further understanding from CDK

- Note that $\mathbf{u} = \bar{\mathbf{u}} + \underline{\mathbf{u}}$ = mean flow + fluctuation.
- This motivates us to reformulate the system for sub-solutions as

$$\rho_t + \mathrm{div}(\rho\bar{\mathbf{u}}) = 0,$$
$$(\rho\bar{\mathbf{u}})_t + \mathrm{div}(\rho\bar{\mathbf{u}} \otimes \bar{\mathbf{u}} + \bar{P}I_n + \rho R) = 0.$$

where the Reynolds stress

$$R = \overline{\mathbf{u} \otimes \mathbf{u}} - \bar{\mathbf{u}} \otimes \bar{\mathbf{u}} + (\overline{\rho^\gamma} - \bar{\rho}^\gamma)I_n$$

is symmetric and positive semidefinite.

Main result

Theorem (Chen-Vasseur-Y. , Adv. Math, 2021)

Whenever $1 < \gamma \leq 1 + \frac{2}{n}$*, for any* $\varepsilon > 0$ *and any* (ϱ^0, U^0) *such that* $E(\varrho^0, U^0) \in L^1(\mathbb{T}^n)$*, there exist infinitely many* (ρ^0, V^0) *satisfying*

$$\rho^0 > 0, \quad E(\rho^0, V^0) \in L^1(\mathbb{T}^n),$$

$$\|\rho^0 - \varrho^0\|^{\gamma}_{L^{\gamma}(\mathbb{T}^n)} + \left\| \frac{V^0}{\sqrt{\rho^0}} - \frac{U^0}{\sqrt{\varrho^0}} \right\|^2_{L^2(\mathbb{T}^n)} < \varepsilon,$$

such that, for each of such initial values (ρ^0, V^0)*, there exist infinitely many global admissible weak solutions* (ρ, V) *to the isentropic Euler equations.*

Remarks: ∞ many solutions

- The most interesting range of γ in physics is $1 < \gamma \leq \frac{5}{3}$ in 3D.
- This result can be regarded as a compressible counterpart of the one obtained by Szekelyhidi–Wiedemann (ARMA, 2012) for incompressible flows.
- The admissibility condition is defined in its integral form. In particular, the energy is decreasing in time t.
- The energy equality could be hold under particular conditions, see
  Y.(ARMA,2017), R. Chen-Y.(JMPA,2019),
  Akramov-Debiec-Skipper-Wiedemann (Anal. PDE, 2020),
  Feireisl-Gwiazda-Swierczewska-Gwiazda-Wiedemann(ARMA,2017)
  ....

Key steps

- Two steps: the construction of *subsolutions*, and the convex integration of these subsolutions to obtain actual solutions.
- Can we construct a sub-solution as follows

$$\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0,$$
$$(\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u} + P(\rho) I_n + \rho R) = 0?$$

  - Vanishing viscosity limits from the Navier-Stokes equation.
  - Weak limits for nonlinear term can produce R.
- We need a suitable convex integral tool?
  - a topological Bairé category argument.
- The energy-compatible subsolution (ρ, V, R), denoting $U := (V \otimes V - Id|V|^2/n)/\rho$, the oscillatory perturbations $(\widetilde{V}, \widetilde{U})$, readily generate $(\rho, V + \widetilde{V})$ as solutions to the the isentropic Euler system.

Existence of NS

Proposition

Fo any $\gamma > 1$, there exists the global weak solution (ρ_ν, V_ν) to

$$\begin{cases} \partial_t \rho_\nu + \operatorname{div} V_\nu = 0, \\ \partial_t V_\nu + \operatorname{div}\left(\dfrac{V_\nu \otimes V_\nu}{\rho_\nu} + p(\rho) I_n\right) = \operatorname{div}\left(\sqrt{\nu\rho_\nu}\,\mathbb{S}_\nu\right), \end{cases}$$

where $\sqrt{\nu\rho_\nu}\,\mathbb{S}_\nu := \nu\rho_\nu \mathbb{D}v_\nu$ *with* $\mathbb{D}v_\nu := \left(\frac{\nabla v_\nu + \nabla^T v_\nu}{2}\right)$ *and* $V_\nu = \rho_\nu v_\nu$.

- This weak solution was constructed by Vasseur-Y. and Bresch-Vasseur-Y. .
- The standard theory need $\gamma > \frac{3}{2}$ in the framework of Lions-Feireisl.
- The most interesting range of γ in physics is $1 \le \gamma \le \frac{5}{3}$.

Vanishing viscosity limits

- As $\nu \to 0$, up to a subsequence,

$$(\rho_\nu, V_\nu) \rightharpoonup (\rho, V) \ \text{ weakly in } \ L^\infty(\mathbb{R}_+; L^\gamma(\mathbb{T}^n)) \times L^\infty(\mathbb{R}_+; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)),$$

  which defines

$$R := \lim_{\nu\to 0} \frac{V_\nu \otimes V_\nu}{\rho_\nu} - \frac{V\otimes V}{\rho}, \qquad r := \lim_{\nu\to 0} p(\rho_\nu) - p(\rho) \quad \text{in } \mathscr{D}'.$$

- $\frac{|V_\nu|^2}{\rho_\nu} \rightharpoonup \frac{|V|^2}{\rho} + \mathrm{Tr}R$, $P(\rho_\nu) \rightharpoonup P(\rho) + r$, by energy inequality, we have

$$\int_{\mathbb{T}^n} \left(E(\rho, V) + \frac{1}{2}\mathrm{Tr}R + \frac{r}{\gamma - 1} \right) dx \le E_0.$$

- Then there exist a subsolution (ρ, V, R, r) of the compressible Euler equations with energy inequality, called $(\mathscr{E}^0, T)$-energy compatible subsolution.

Regularity and positivity enhancement

- Subsolutions via v.v. are rough and $R + rI_n$ may degenerate.

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Regularity and positivity enhancement

- Subsolutions via v.v. are rough and $R+rI_n$ may degenerate.
- Smoothing via convolution.
- Enhancing positivity via convex combination of $(\mathscr{E}^0, T)$-energy compatible subsolutions.
- The above two procedures respect energy compatibility because of convexity.
- Therefore we are left to consider convex integration from smooth energy compatible subsolutions with positive definite total defect matrix $R+rI_n$.

Oscillation lemma

Proposition (***Chen-Vasseur-Y., 2021.***)

There exist infinitely many $\widetilde{V}$ *and traceless* $\widetilde{U}$ *(as oscillatory perturbations), both supported in* Ω*, such that in* $\mathbb{R}^n \times \mathbb{R}_+$*:*

$$\begin{cases} \operatorname{div} \widetilde{V} = 0, \\ \partial_t \widetilde{V} + \operatorname{div} \widetilde{U} = 0, \end{cases}$$

while

$$\frac{(V+\widetilde{V}) \otimes (V+\widetilde{V})}{\rho} - (U+\widetilde{U}) = \left(\frac{|V|^2}{n\rho} + q \right) I_n$$

is achieved as to eliminate the Reynolds stress $R := qI_n$*.*

Energy injection

$(\rho, V+\tilde{V})$ Euler solution.

$$\frac{|V+\tilde{V}|^2}{\rho} = \frac{|V|^2}{\rho} + \operatorname{tr} R.$$

$\frac{1}{2}\operatorname{tr} R$ is pumped into the kinetic energy density through C.I..

- The subsolutions

$$\begin{cases} \partial_t \rho + \operatorname{div} V = 0, \\ \partial_t V + \operatorname{div}\left(\dfrac{V \otimes V}{\rho} + p(\rho) Id + R\right) = 0. \end{cases}$$

- There exist infinitely many $\widetilde{V}$ and traceless $\widetilde{U}$ (as oscillatory perturbations):

$$\begin{cases} \operatorname{div} \widetilde{V} = 0, \\ \partial_t \widetilde{V} + \operatorname{div} \widetilde{U} = 0, \end{cases}$$

  while

$$\frac{(V + \widetilde{V}) \otimes (V + \widetilde{V})}{\rho} - (U + \widetilde{U}) = \left(\frac{|V|^2}{n\rho} + q\right) Id,$$

  is achieved as to eliminate the Reynolds stress $R := qId$.

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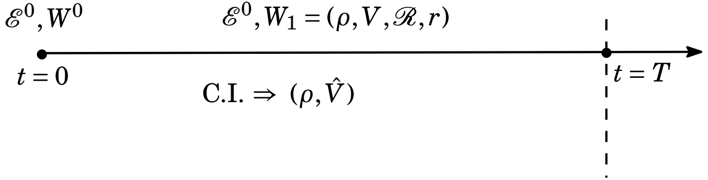
Compensation for potential energy

$\mathscr{E}^0, W^0 = (\rho^0, V^0, \mathscr{R}^0, r^0) \Rightarrow (\mathscr{E}^0, T)$-compatible subsolution W_1

Say $\rho > 0, \quad R = \mathscr{R} + r\mathrm{I}_n > 0$



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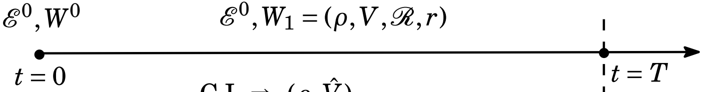

C.I. $\Rightarrow$ $(\rho, \hat{V})$

Energy $\quad E(\rho, \hat{V}) = E(\rho, V) + \frac{1}{2}\mathrm{tr}R$

$$= E(\rho, V) + \frac{1}{2}\mathrm{tr}\mathscr{R} + \frac{n}{2}r$$

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$\mathscr{E}^0, W^0$

$\mathscr{E}^0, W_1 = (\rho, V, \mathscr{R}, r)$

$t = 0$

$t = T$

C.I. $\Rightarrow (\rho, \hat{V})$

Energy $E(\rho, \hat{V}) = E(\rho, V) + \frac{1}{2}\mathrm{tr}R$

$= E(\rho, V) + \frac{1}{2}\mathrm{tr}\mathscr{R} + \frac{n}{2}r$

Energy $\neq$: $E(\rho, \hat{V})$ v.s. $E(\rho, V) + \frac{1}{2}\mathrm{tr}\mathscr{R} + \frac{r}{\gamma - 1}$

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$\mathscr{E}^0, W^0$ $\qquad$ $\mathscr{E}^0, W_1 = (\rho, V, \mathscr{R}, r)$

$t = 0$ $\qquad\qquad$ $t = T$

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$$1 < \gamma \leq 1 + \frac{2}{n}$$

$\Rightarrow$ need compensation for

potential energy density

Compensation for potential energy

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$\mathscr{E}^0, W^0$ $\qquad$ $\mathscr{E}^0, W_1 = (\rho, V, \mathscr{R}, r)$

$t = 0$ $\qquad$ $t = T$

$$r_c(t) = \left(\frac{2}{n(\gamma-1)|\mathbb{T}^n|}\right)\int_{\mathbb{T}^n} r(t,x)\,dx$$

Consider $(\mathscr{E}^0, T)$-compatible subsolution

$$W = (\rho, V, \mathscr{R}, r + r_c)$$

Compensation for potential energy

$$\mathscr{E}^0, W^0 = (\rho^0, V^0, \mathscr{R}^0, r^0) \Rightarrow (\mathscr{E}^0, T)\text{-compatible subsolution } W_1$$

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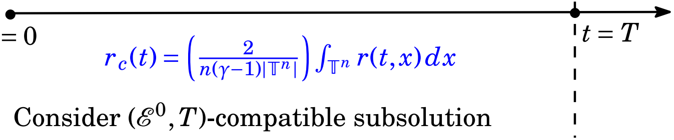

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Issue: bump-up of initial energy

$$E(\rho^0, V^0) \longrightarrow E(\rho^0, V^0) + \tfrac{1}{2}\mathrm{tr}R$$

Double convex integration

$$\mathscr{E}^0, W^0 = (\rho^0, V^0, \mathscr{R}^0, r^0) \Rightarrow W = (\rho, V, \mathscr{R}, r + r_c)$$

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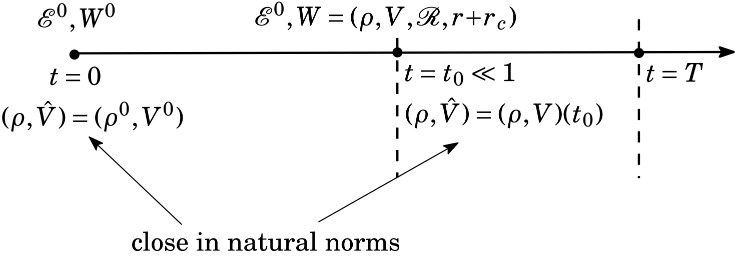

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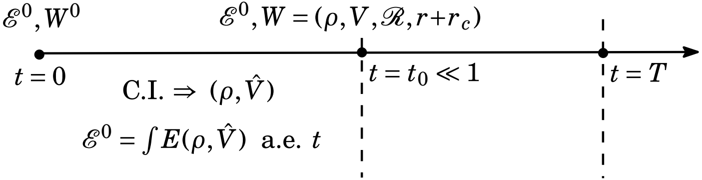

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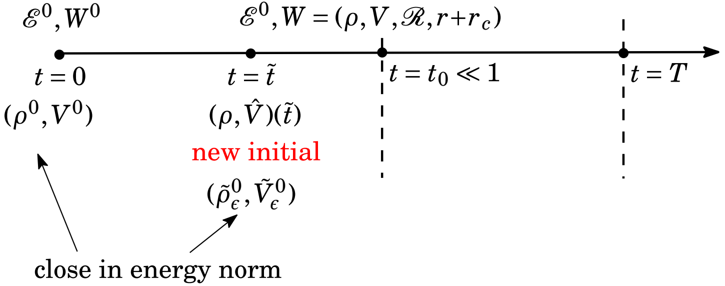

$$\mathscr{E}^0 = \int E(\tilde{\rho}^0_\epsilon, \tilde{V}^0_\epsilon)$$

Double convex integration

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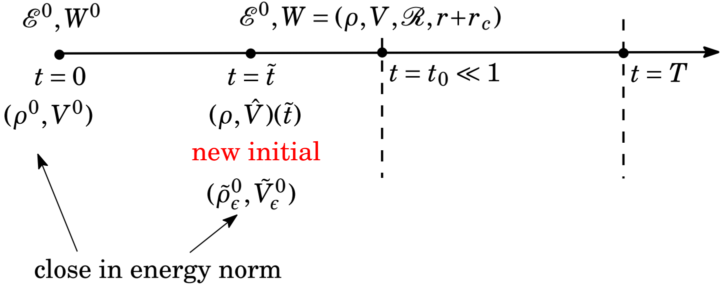

$$\mathscr{E}^0 = \int E(\tilde{\rho}^0_\epsilon, \tilde{V}^0_\epsilon)$$

Note: ∞ many choices for $\tilde{t} \Rightarrow \infty$ many initial data $(\tilde{\rho}^0_\epsilon, \tilde{V}^0_\epsilon)$

Double convex integration

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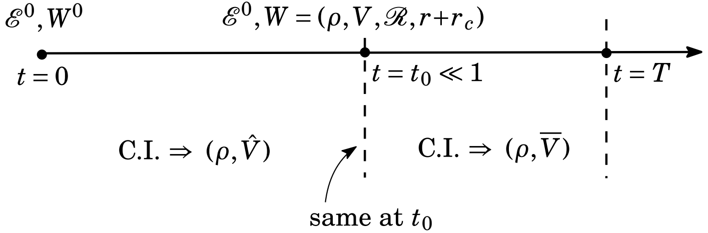

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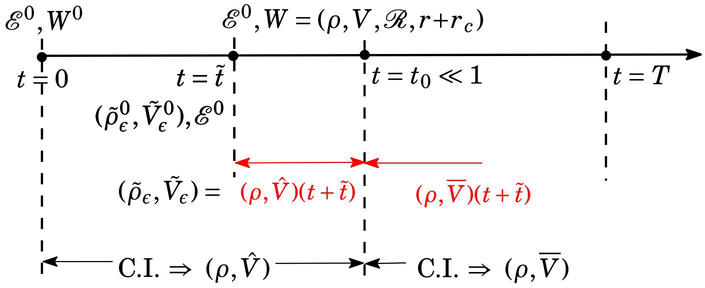


Thank You!