Confirmation of Status Report

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Abstract

Gaussian fields are ubiquitous in probability as they are scaling limits of many natural objects, and in applied science as they are very useful in modeling natural phenomenon. In this document, we study geometric features of smooth Gaussian fields, like measure of level sets and structure of critical points of the field. Large scale geometry, i.e. studying geometric observables in a domain where the domain size goes to infinity, is of particular interest.

In the first chapter, we give some background on smooth Gaussian fields, including motivation. We formally define smooth Gaussian fields, state basic properties like existence using Kolmogorov's theorem, Bochner's theorem etc. Then we briefly explain connections to other topics in mathematics including percolation theory, quantum chaos, real algebraic geometry.

Next, we study measure of level sets of stationary Gaussian fields. This quantity has been researched extensively in the last 20 years, using tools like Kac-Rice formula, Wiener chaos expansion [Wig22]. We now have a good understanding of expectation, fluctuation in specific models like random harmonics on sphere. In the second chapter, we ask, given two Gaussian fields which are coupled closely, how close are their measures of level sets in a domain. Here we bring in novel ideas (in this context) from geometric analysis to answer this question. We prove convergence of Hausdorff measure of level sets of smooth Gaussian fields when the levels converge. Given two coupled stationary fields f_1, f_2 , we estimate the difference of Hausdorff measure of level sets in expectation, in terms of C^2-fluctuations of the field $F = f_1 - f_2$. The main idea in the proof is to represent difference in volume as an integral of mean curvature using the divergence theorem. This approach is different from using Kac-Rice type formula as main tool in the analysis. This chapter is based on joint work with Dmitry [BH23].

Now, in third chapter we study critical point structure of smooth Gaussian fields. Critical points of smooth fields give important information on landscape of the field, and on topology of level sets. For example, topology of level sets remain unchanged between two levels if there's no critical point in the middle, thanks to Morse theory. In applied sciences like astronomy, medical imaging, extrema and critical points are readily observed, hence crucial to analyse them. In this article, we consider the point process of local maxima above a level $u(R)$ in a growing region $[0, R]^d$. We show that this point process converges weakly to Poisson point process in the limit $R \to \infty$. In the literature, high excursion sets of many smooth and non-smooth Gaussian process of decaying correlation are well studied [LLR83]. Also, for Gaussian processes with some Markov property like Brownian motion, Gaussian free field, high points (after suitable rescaling) have been shown to converge to Poisson process. But in all processes considered, to the best of our knowledge, the threshold level is

comparable to that of expected maxima in the region. Here, we show that, for *any* arbitrary threshold $u(R) \to \infty$, we observe Poisson process in the weak limit. Proof relies on the classical observation that simple point processes are characterised by avoidance probabilities (i.e. $\mathbb{P}(\eta(B) = 0)$ for Borel sets B). Then we approximate avoidance probability with excursion probability, where the latter is well studied.

Last but not the least, appendix contains brief info on the standard tools in this topic, like Kac-Rice formula. Part A of the appendix contains many key lemmas and theorem which are used multiple times in chapter 3.

Contents

Chapter 1

Preliminaries on smooth Gaussian fields

1.1 Motivation

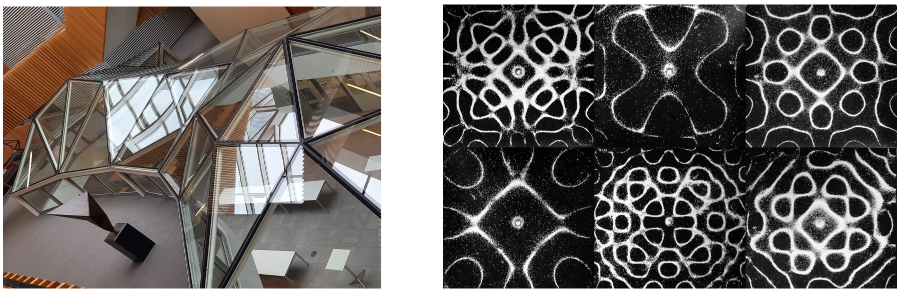
At the end of the 18th century, the musician and physicist Chladni noticed that sounds of different pitch could be made by exciting a metal plate with the bow of a violin, depending on where the bow touched the plate. The latter was fixed only in the center, and when there was some sand on it, for each pitch a curious pattern appeared. Some years later, it was realised that Chladni figures correspond to zeros of eigenpairs (eigenvalues and corresponding eigenfunctions) of a wave operator. One can show that this problem can be quickly relegated to the study of Laplace eigenfunctions.

Studying the Laplace eigenfunctions and their geometry is a classical subject and of great interest to both mathematicians and physicists, going back to at least 19th century. Advent of quantum mechanics in 20th century fueled this area of mathematics further. Much of the quantum mechanics is concerned with the eigenvalue problem for the Schrodinger equation,

$$\left(-\frac{h^2}{2}\Delta + V\right)\psi = E\psi \tag{1.1}$$

where h is Planck's constant, V is the potential, ψ is the wave-function, E energy level of the wave-function. Many important questions about these involve studying behaviour of "typical" eigenfunctions, where we are lead to Gaussian random fields. Examples of some of the models considered in this context include Random Plane Waves (RPW) (which are random eigenfunctions of laplacian in R^2), Gaussian random linear combination of deterministic eigenfunctions.

Apart from this, Gaussian fields appear naturally in many practical applications. Especially when one encounters random surfaces in real life, like photographs, television pictures, topographic maps, atmospheric pressure charts,



(a) South crystal in Oxford maths institute

(b) Chladni figures

Figure 1.1: (a) The crystal is a triangulation of the amplitude of the first overtone of that domain. (b) Sand accumulating on nodes corresponding to different resonating frequencies

studying statistical properties of the contours the surface is helpful [Swe62]. Random fields have found some applications in areas as diverse as oceanography [LH57], cosmology [BBKS86], medical imaging [WMN$^+$96].

1.2 Smooth Gaussian fields

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $V \subset \mathbb{R}^n$ be open set. A function $f : V \times \Omega \to \mathbb{R}$ is called a Gaussian function (more commonly, a Gaussian field) if

1. for each $x \in V$, the mapping $\omega \to f(x, \omega)$ is measurable as a mapping from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$;
2. for each finite set of points $x_1, x_2, \ldots x_n \in V$ and for each $c_1, \ldots c_n \in \mathbb{R}$, the sum $\sum_j c_j f(x_j, \omega)$ is a Gaussian r.v. (being degenerate is allowed).

Let $k \in \mathbb{N}$. The Gaussian function f is called C^k-smooth if

3. for almost every $\omega \in \Omega$, the function $x \to f(x, \omega)$ belongs to the space $C^k(V)$.

Given a Gaussian field $f : V \to \mathbb{R}$ (which is a common abuse of notation), define its *covariance kernel* to be $K(x, y) := \mathbb{E}[f(x)f(y)]$ for $x, y \in V$.

Theorem 1.2.1 (Kolmogorov's theorem, [NS16])**.** *Let $k \in \mathbb{N}$. Suppose that $K : V \times V \to \mathbb{R}$ is a positive definite symmetric function of class $C^{k,k}(V \times V)$ and, in addition, that*

$$\max_{|\alpha|,|\beta| \leq k} \sup_{x,y \in V} |\partial_x^\alpha \partial_y^\beta K(x, y)| < \infty.$$

Then there exists a (unique up to an equivalence of distribution) C^{k-1} Gaussian function f on V with the covariance kernel K.

This says, if we have a 'nice/smooth' covariance kernel then we'll get 'nice/smooth' enough field. We have plenty of examples of (interesting) positive definite functions, such as the Gaussian kernel (e^{-ax^2}), Sinc kernel ($\sin x/x$) etc. Without loss of generality, we assume that *all* Gaussian fields in this article are centred (i.e. $\mathbb{E}f(x) = 0, \forall x$) because we can subtract a deterministic function to get another Gaussian field.

We describe three ways to think about Gaussian fields. The first one is more 'probabilistic' way. A Gaussian field is a stochastic process on an index set V, where the random variables $f_x = f(x), x \in V$ are jointly Gaussian. We know that, thanks to Kolmogorov's theorem, such processes are determined by pairwise covariances $K(x,y) = \mathbb{E}[f(x)f(y)]$. While it seems that the nature/topology of the index set is irrelevant from this viewpoint, when we demand nicer structure on the field it becomes important.

A more 'analytic' approach to Gaussian fields in the following. We can think of $f(x)$ as function drawn at random from some space of functions. More or less equivalently, it can be also thought as a random variable corresponding to a Gaussian measure on some function space. One canonical (and simplest) way to 'write down' a Gaussian function is a random linear combination of deterministic functions, usually written as

$$f(x) = \sum a_i \psi_i(x)$$

where a_i's a re i.i.d. standard normal variables and $\{\psi_i(x)\}$ are orthonormal basis of that function space.

We can switch from one viewpoint to the other in many cases. Consider $f = \sum a_i \psi_i$ as a formal series and compute $\mathbb{E}[f(x)f(y)]$ formally. We get,

$$K(x,y) = \sum \psi_i(x)\psi_i(y).$$

The other direction is bit more involved. Call a continuous Gaussian field on $\mathbb{R}^n$ *stationary* or *translation invariant* if its covariance kernel $K(x,y)$ depends only on $x - y$, say $K(x,y) = \kappa(x-y)$. By Bochner's theorem, κ can be written as a Fourier transform of some finite symmetric (means $\mu(-A) = \mu(A)$) positive Borel measure measure ρ on $\mathbb{R}^n$, i.e.,

$$\kappa(x) = \int_{\mathbb{R}^n} e^{2\pi i(\lambda \cdot x)} d\rho(\lambda).$$

Call the measure ρ *spectral measure of the field f.* Thinking of a Gaussian field in terms of its spectral measure is more geometric while that of kernels are more analytic. And most important examples of Gaussian fields that we're interested in are stationary anyways. Now the field is given by the Fourier transform of the white noise on the spectral measure. One description of white noise on the spectral measure is the following.

Consider the Hilbert space $L^2_{sym}(\rho) = \{\psi \in L^2(\rho) : \psi(-t) = \bar{\psi(t)}\}$. Now, the Fourier transform of symmetric functions will be real valued, and forms a Hilbert space, call it $H = \mathcal{F}L^2_{sym}(\rho)$ with inner product inherited from L^2 space. A white noise W on the measure ρ (See appendix) would be an isonormal process on H, i.e. an inner product preserving map from H taking values in a Gaussian Hilbert space. So our field has a representation,

$$f(x) = W(e^{2\pi i x\cdot t}).$$

Now by considering orthonormal basis of H (which are Fourier transforms of an orthonormal basis of $L^2_{sym}(\rho)$), we can recover 'analytic' picture of the field, at least on formal level.

To think of the field in terms of spectral measure, let's start with a simple example. When the spectral measure is a two point measure, the field will be a sine wave in the direction of line passing through these points with a Gaussian amplitude, variance being the measure of these points and wavelength as inverse of distance from origin. That is, if $\mu = (\delta_t + \delta_{-t})/2$ where $t \in \mathbb{R}^2$ then corresponding field is $f(x) = A\cos(2\pi x \cdot t)$. Now if we have a spectral measure supported on $2n$ points, then the field will be a random super-imposition of sine waves in the direction of these points with amplitudes' variance proportional to the measure at those points. In [BM22], Belyaev and Maffucci came up with a coupling of fields which are close when the spectral measures are close, with high probability. So we can approximate fields by considering approximations of spectral measures, which gives us more geometric picture at times.

Please refer to [AT09], [AW09], and appendix of [NS16] for more on smooth Gaussian fields.

1.3 Connections to other topics in maths

Now we describe very briefly some topics in maths which are related to smooth Gaussian fields.

Percolation theory: Let us first describe percolation theory briefly. One of the simplest and non-trivial percolation model is Bernoulli bond percolation. Consider the nearest neighbor graph of the planar lattice $\mathbb{Z}^2$. Remove every edge with probability $p \in [0, 1]$, independently of each other. We are interested in the large scale connectivity property of the random graph thus obtained. One of the first questions we ask about a percolation model is whether there is an infinite cluster. Let $\theta(p)$ be the probability that, say, the origin is in an infinite cluster and define $p_c := \inf_{p\in[0,1]}\{p : \theta(p) > 0\}$, the critical probability.

Many natural questions crop up immediately, some of them being: 1) Is p_c non-trivial? 2) Is there an infinite cluster at $p = p_c$? 3) If $0 < p_c < 1$, how does the phase transition occurs? 4) When $\theta(p) = 0$, what is the typical size of clusters?
All four of the above questions are well understood in the case of planar Bernoulli bond percolation, thanks to the analysis available from considering dual graphs. But some of the more sophisticated questions are yet to be fully answered, like is scaling limit of this model (if exists) conformally invariant at $p = p_c$? For example, when you take the mesh size of the lattice to zero, are crossing events conformally invariant? It is conjectured that the critical percolation is conformally invariant in the scaling limit, independent of the planar lattice structure. Smirnov, in his celebrated paper [Smi01], answered it for triangular lattices.

We can ask similar questions as above in the context of smooth Gaussian fields. The tools/techniques used in discrete models of percolation often translate directly or have an analogy in this setting, of course presenting lot of additional challenge sometime. A notable conjecture in this regard is one made by Bogomolny and Schmidt in [BS02], which offers a bond percolation model to random plane waves. They argued that *nodal lines* (i.e. the zero set of the field) form a square lattice, but since the nodal lines do not intersect almost surely. So the supposed 'ties' can be resolved one or an another equally likely, due to symmetry of the field. This paints a critical percolation picture of the random planar waves. Some argued that the numericals suggested were a bit off [BK13].

The following form of the Bogomolny-Schmidt conjecture is believed to be true:

Conjecture 1.3.1. *All large scale connectivity properties of the RPW nodal lines and domains are the same as for the critical percolation. In particular, all crossing events have the same scaling limits. The collection of all nodal lines has a scaling limit which is conformally invariant and same that of critical percolation model.*

An extended version of this conjecture states that for very general class of smooth Gaussian fields, obeying some regularity conditions, the the nodal domains of the field should have the same large scale properties as critical percolation clusters and excursion sets for non-zero levels should behave like off-critical percolation clusters.

Refer [DC], [Gri99] and references therein for more percolation theory. See [Bel22] for a survey on the relation between percolation theory and smooth Gaussian fields.

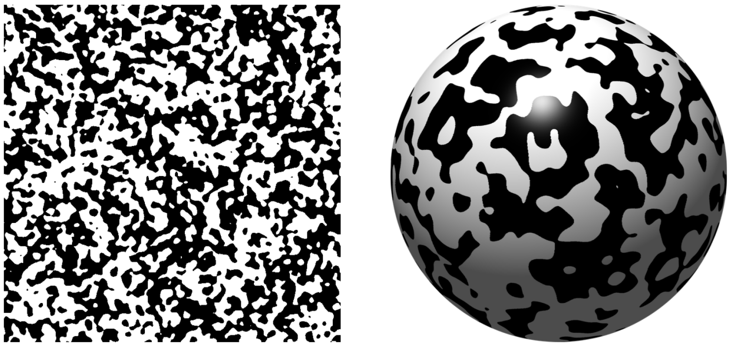


Figure 1.2: (Left) Bargman-Fock field sample. (Right) Gaussian ensemble of homogeneous polynomials of degree 300. The scale is $d^{-1/2}$ where d is the degree. Picture: Dmitry Beliaev

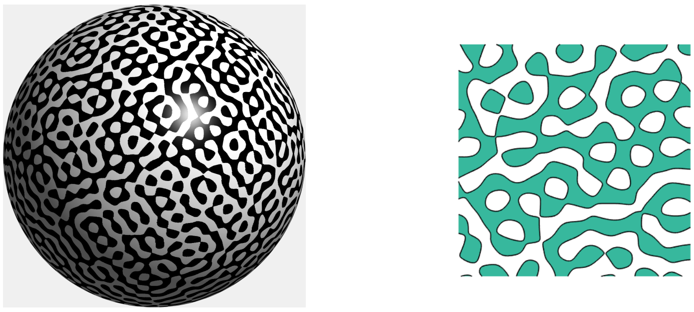
Real algebraic geometry: The first part of Hilbert's 16th problem asks to analyse number of components and arrangement of an algebraic curve of degree n. Real algebraic hyperspaces in projective spaces are well studied and it is interesting to investigate 'typical' behaviour of these objects. Kostlan ensembles, which is a Gaussian measure on the space of homogeneous polynomials of degree-d (say), is one of the natural objects to consider. An important case is the behaviour when the degree is large. Interesting percolation theoretic properties, like RSW estimates, has been observed in this model [BMW17]. Also, Kostlan ensemble has a translation invariant local scaling limit as $d \to \infty$, called the Bargmann-Fock field.

Let's define Bargmann-Fock fieldon $\mathbb{R}^2$ as follows

$$F(x) = F(x_1, x_2) = e^{-|x|^2/2} \sum_{m,n=0}^{\infty} \frac{a_{m,n}}{\sqrt{m!n!}} x_1^m x_2^n \tag{1.2}$$

Now, this field is a scaling limit of many such Gaussian fields, especially the ones with algebraic origins. Its spectral measure is the Gaussian measure on $\mathbb{R}^2$, hence the covariance kernel has super-exponential decay at infinity (or can be computed directly, $\mathbb{E}[F(x)F(y)] = e^{-|x-y|^2/2}$). Due to this rapid decay, it is sometimes amenable to percolation theoretic methods.

Quantum Chaos: Dynamics of a classical particle in a bounded domain (either with some potential or no potential with reflective boundaries) is a well studied topic in maths and physics which goes back to at least to Newtonian era. Stability (or lack of) of these particle was of particular interest in dynamical systems. Consider a simple model system, that of a billiard particle. The description of the system in the language of *classical mechanics* is of a point particle moving without friction in a billiard table - a bounded planar enclosure where the particle reflects from the boundary so that the angle of incidence equals equals the angle of reflection. Now, a *quantum mechanical*



(a) Random spherical harmonic of high degree

(b) A closer look at random plane wave nodal lines

Figure 1.3: Both pictures by Dmitry Beliaev

description of this system at a given instant of time includes the wave function of the particle $\psi(x,t)$ which vanishes at the boundary of the billiard.

We can relate the two descriptions by taking $V = 0, E = 1, h = \lambda^{-1}$ and letting $h \to 0$ in equation (1.1), sometimes called the semiclassical limit. In 1977 [Ber77], Berry suggested that classical behavior of the particle on a generic billiard is characterised by local behaviour of wavefunctions in the semiclassical limit. Specifically, in the chaotic case wavefunctions behave 'locally'(at a certain scale) as a uniform random superimposition of monochromatic waves in all directions, random plane wave model (RPW) as we call it now.

The RPW has the following series representation,

$$F(x) = \sum_{\infty}^{\infty} C_n J_{|n|}(|x|)e^{in\theta} \tag{1.3}$$

where C_k are independent standard complex Gaussian random variables satisfying $C_{-k} = \bar{C}_k$ and J_k is the kth Bessel function. The covariance kernel turns out to be $J_0(|x|)$, which is an oscillating function and decays like $|x|^{-1/2}$. Now the corresponding spectral measure is the normalised arc-length on the unit circle in $\mathbb{R}^2$. Hence sample functions are eigenfunctions of the Laplacian in the plane, almost surely. Also, as mentioned above, intuitively the field is a random interference of monochromatic waves, uniform in all direction. RPW also appears as a scaling limit of a variety of fields, for example 'band-limited' functions [BW16].

In the last two decades, there has been numerous suggestions of observables for the quantum chaos classification problem. Blum, Gnutzmann and Smilansky [BGS02] suggested that nodal domain count distribution as a criterion. Bogomonly and Schmidt [BS02] proposed a percolation model for the RPW model which is the conjectured local behaviour of chaotic systems. It is also worth noting that some of the observable suggested, say eigenvalue gaps, are related to that of random matrix theory.

Helmholtz equation: Consider $(\mathcal{M}, g)$ a closed Riemannian manifold of dimension 2 and a smooth function $f : \mathcal{M} \to \mathbb{R}$. Let Δ denote the Laplace-Beltrami operator on the manifold. The geometry of eigenfunctions of Δ is great importance in the analysis of PDEs on manifolds. Also the geometry of these manifolds is related (conjecturally in some cases) to the eigenvalue problem, say of minimal embedded hyper-surfaces [LY82]. One of the notable conjectures of Yau states that length of nodal lines of the eigenfunctions is comparable to square root of the corresponding eigenvalues when they're large. Specifically, $\exists\ c_{\mathcal{M}}, C_{\mathcal{M}} > 0$ such that

$$c_{\mathcal{M}}\sqrt{\lambda_j} \leq \text{length}(f_j^{-1}(0)) \leq C_{\mathcal{M}}\sqrt{\lambda_j} \tag{1.4}$$

where $(f_j, \lambda_j), j \geq 1$ are eigenpairs of Δ. See [LM19] for a survey on progress on Yau's conjecture. We can ask whether similar estimates as (1.4) holds on an average for various Gaussian fields model, for example random band-limited function model.

Arithmetic random waves: The study of Laplace eigenvalues and eigenfunctions on 2-torus is linked to that of lattice points on ellipses/circles in classical number theory. Let us define the random Gaussian Laplace toral eigenfunctions as follows. Consider the the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Now the eigenvalues are of the form $E_m = 4\pi^2 m$ where m is a sum of two integer squares. Let

$$\Lambda = \Lambda_m = \{\lambda \in \mathbb{Z}^2 : |\lambda|^2 = m\}$$

be the set of lattice points on the circle $\sqrt{m}\mathbb{S}^1$. For the eigenvalue E_m the collection of exponentials

$$\{e^{2\pi i \langle \lambda, x\rangle}\}_{\lambda \in \Lambda_m}$$

forms a basis for the eigenspace. Consider the following (rescaled) Gaussian ensemble of eigenfunctions,

$$\psi_m(x) = \frac{1}{r_2(m)} \sum_{\lambda \in \Lambda_m} a_\lambda e^{2\pi i \langle \lambda/m, x\rangle}, x \in \sqrt{m}\mathbb{T}^2$$

where $r_2(m) = |\Lambda_m|$ and a_λ is complex Gaussian with unit variance. Now the corresponding spectral measure is,

$$\nu_m = \frac{1}{r_2(m)} \sum_{\lambda \in \Lambda_m} \delta_{\lambda/\sqrt{m}}$$

where δ_x is the Dirac measure on the point x. Note that this measure is supported on $1/\sqrt{m}\Lambda_m \subset \mathbb{S}^1$. This family of spectral measures changes in a complicated way and does not converge as $m \to \infty$. However, along a generic (i.e. density one) subsequence, it converges weak-* to the uniform measure on $\mathbb{S}^1$. To the other extreme, there are subsequences (of zero density) along where the the measure converges in weak-* to $1/4(\delta_{\pm 1} + \delta_{\pm i})$, called the Cilleruelo measure. Cilleruelo-type fields serves as a motivation for us to study degenerate fields in more detail.

Chapter 2

Measure of level sets

2.1 Introduction

Studying geometrical and topological properties of the field, especially of level/ excursion sets of the field is of great interest. Particularly, functionals such as volume of level sets, number of connected components of level sets are well studied (see [Wig22],[NS16]). In problems involving Gaussian fields, sometimes one needs to compare two fields, say by coupling them, when their laws are close. Comparing geometric observables are of particular interest. We show that, with probability close to one, difference in Hausdorff measures of *nodal sets* (i.e. the zero sets) of coupled fields with 'close' laws is small. The main idea in the proof is to represent difference in volumes of level sets as an integral of mean curvature of the hypersurface using the divergence theorem. This representation is classical in Riemannian geometry and has been used extensively in study of minimal surfaces [Law80, Chapter 1]. The novelty is to get an average estimate of the difference in volumes in the context of Gaussian fields. Also, we don't rely on Kac-Rice (or any other variation of co-area formula) for the analysis of volume of level sets, which is a standard tool in this topic. As a by product, we give an explicit formula for the mean curvature of level sets at a given level. We believe that proving convergence in distribution of Hausdorff measure of level sets can be done by following the proof idea of Kac-Rice as presented in [AW09, Theorem 6.2]. But it might not be as straight forward as our proof, and proving other convergences might require some new ideas.

2.2 Results

In this article, we consider smooth Gaussian fields $f : \mathbb{R}^d \to \mathbb{R}$ with mild non-degeneracy conditions, of fixed dimension $d \geq 2$. Call a field *stationary* if the covariance kernel $K(x, y) = \mathbb{E}[f(x)f(y)]$ is translation invariant. Now for stationary fields, the kernel K is a Fourier transform of a finite symmetric Borel measure ρ, called spectral measure.

Fix a domain $D = [-R, R]^d \subset \mathbb{R}^d$. Consider two C^2-smooth Gaussian fields $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}$ and a coupling of the fields f_1, f_2, by abuse of notation, such that $F = f_1 - f_2$ has the C^2-fluctuations

$$\sigma_D^2 := \sup_{x \in D} \sup_{|\alpha| \leq 2} \mathrm{Var}[\partial^\alpha F(x)].$$

Assumptions 2.2.1. *Assume that the fields f_1, f_2 are*

1. *stationary, C^2-smooth a.s.*
2. *non-degenerate, i.e. $(f_i, \nabla f_i)$ has density in $\mathbb{R}^{d+1}$ for $i = 1, 2$.*
3. *Morse functions a.s.*

Let $\mathcal{L}^n$ denote n-dimensional Lebesgue measure. Let $\mathcal{H}^n$ denote the n- dimensional Hausdorff measure, which is scaled so that $\mathcal{H}^n([0,1]^n) = \mathcal{L}^n([0,1]^n)$. Note that by Bulinskaya lemma (see [NS16, section 5.3]), a.s. nodal sets are sub-manifolds of $\mathbb{R}^d$ of co-dimension one. So we interchangeably use the terms volume and Hausdorff measure.

Theorem 2.2.2. *Let $\mathcal{H}^{d-1}(f_i^{-1}(a))$ denote the volume of level sets in the domain D. With the setup as above, we have*

$$\mathbb{E}|\mathcal{H}^{d-1}(f_1^{-1}(0)) - \mathcal{H}^{d-1}(f_2^{-1}(0))| \leq C(f_1, f_2)(\mathcal{L}^d(D)\sqrt{\log R})\sigma_D^{1/7}$$

assuming σ_D is small enough (say, $\sigma_D < 1$). Here, the constant $C(f_1, f_2)$ depends only on the laws of the fields and not the coupling.

The factor $\sqrt{\log R}$ appearing in the above theorem is from the quantitative version of Kolmogorov's existence theorem for smooth Gaussian fields as stated in [NS16, Appendix A]. Also, the exponent $1/7$ in $\sigma_D^{1/7}$ is not optimal, and can be made close to $1/4$ in the proof. We believe optimal exponent of σ_D is 1 due to cancellations in the integral of mean curvature in the bulk.

We make some comments on the assumptions on the fields. We believe that the proof of Theorem 2.2.2 works for non-stationary fields with positive lower bounds on fluctuations of the field and its derivatives with suitable modifications but computations become tedious. Only the corollary uses the stationarity assumption in a crucial way. Also, assumption that the fields are a.s. Morse functions is not very restrictive and many interesting non-degenerate fields we know are Morse functions a.s. It can be shown that stationary fields with spectral measures containing an open set are Morse a.s. If the field is isotropic, then also we can show that the field is Morse a.s. In particular, random plane wave (RPW) model and Bargmann-Fock field (on $\mathbb{R}^d$) are Morse a.s.

One such coupling of fields is available using coupling of white noises (see [BM22]). The coupling as in [BM22] gives the following estimate for the fluctuations of the field $F = f_1 - f_2$. We have,

$$\sigma_D^2 \leq C(R^d + 1) \inf_{\rho \in \mathcal{P}(\rho_1, \rho_2)} \int (|s|^2 + |t|^2 + 1)^{2+1} |s - t|^2 d\rho(s,t)$$

where $\mathcal{P}(\rho_1, \rho_2)$ is the space of all symmetric couplings of ρ_1 and ρ_2 and C is a absolute constant.
Now by the coupling techniques mentioned above, σ_D can be controlled by the transport distances between the measures in the domain (in the general case) or by norm of differences in spectral densities (in special cases). Let's give an example where this is useful. Recall that the spectral measure of random planar waves is the uniform measure on the unit circle in $\mathbb{R}^2$. We can approximate this measure, in the transport distance mentioned above, by a measure supported on finite points. This field corresponds to a finite interference of pure sine waves. So we can obtain quantitative bounds on the difference of lengths.

To prove Theorem 2.2.2, first we study convergence of volume of level sets using the divergence theorem. Although expressing change in volume of a hypersurface in normal direction in terms of mean curvature is classical as previously mentioned, we need the version as in Proposition 2.2.3.

Proposition 2.2.3. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a non-degenerate, C^2-smooth Gaussian field which is Morse function a.s. Let $\mathcal{H}^{d-1}(f^{-1}(a))$ denote the volume of level set $f^{-1}(a)$ in D. Then, almost surely, we have*

$$\mathcal{H}^{d-1}(f^{-1}(b)) - \mathcal{H}^{d-1}(f^{-1}(a)) = \iint_D \kappa \mathbb{1}_{f \in [a,b]} dvol - \oint_{\partial D} \left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta} \right\rangle \mathbb{1}_{f \in [a,b]} dS \tag{2.1}$$

where

$$\kappa = div \left(\frac{\nabla f}{|\nabla f|} \right)$$

is $(d-1)$ times the mean curvature of level set of f at x and $\hat{\eta}$ is the outward unit normal to the $(d-1)$-dimensional slabs of ∂D. We also have

$$\mathcal{H}^{d-1}(f^{-1}(b)) \to \mathcal{H}^{d-1}(f^{-1}(a)), \text{ as } b \to a$$

almost surely and in L^1.

As a corollary, we get the following formula for the mean curvature of level sets at a given level. Usually, it is hard to get such explicit formula for general fields.

Corollary 2.2.4. *With assumptions as in Theorem 2.2.2 , we have*

$$\mathbb{E}[\kappa | f = a] = -a\mathbb{E}[|\nabla f|].$$

2.3 Proofs

Proof of Proposition 2.2.3. Note that f has only finitely many critical points in D a.s. We prove in subsection 2.3.1 that κ as a function on D is integrable almost surely. We also can assume that f has no critical points on ∂D. This is because of Bulinskaya lemma, since ∂D is $(d-1)$-dimensional and for non-degenerate, smooth Gaussian f the gradient ∇f has (Gaussian) density on $\mathbb{R}^d$.

Case 1: a, b are regular values of f.

Let $R' = D \cap f^{-1}[a,b]$ and the unit outward normal $\hat{\eta} = -\nabla f/|\nabla f|$ on $f^{-1}(a)$, $\hat{\eta} = \nabla f/|\nabla f|$ on $f^{-1}(b)$ (assuming $a < b$), outward normal on parts of $\partial D \cap f^{-1}(a,b)$. Assume that R' has no critical points of f and we know that κ is continuous except at critical points of f. Apply Greens formula for the function $\nabla f/|\nabla f|$ on R', we get

$$\int_{f^{-1}(b)\cap D} \left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta} \right\rangle dS + \int_{f^{-1}(a)\cap D} \left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta} \right\rangle dS + \\ \oint_{\partial D} \left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta} \right\rangle \mathbb{1}_{f\in[a,b]} dS = \iint_{R'} \operatorname{div}\left(\frac{\nabla f}{|\nabla f|}\right) d\mathrm{vol}. \tag{2.2}$$

But first two terms of LHS of above equation are $\mathcal{H}^{d-1}(f^{-1}(b)), -\mathcal{H}^{d-1}(f^{-1}(a))$ respectively. Hence we get equation (2.1) in this case.

If R' has critical points of f, then the number of critical points has to be finite. Let $\{x_1, x_2, \ldots x_k\}$ be the critical points in R'. Now apply the divergence theorem to the field $\nabla f/|\nabla f|$ on $R' \setminus \cup_j B_\delta(x_j)$. Letting $\delta \to 0$, and using integrability of κ on D (see subsection 2.3.1), we again get equation (2.1).

Case 2: a or b (or both) are critical values of f.

First, let us show continuity of volume of level sets at all levels, including at critical values of f. Fix a critical value a of f. By Morse lemma, f can be made a quadratic function at a critical point by re-parametrisation. Let p be a critical point, then there is a neighborhood U of p and a smooth chart $(y_1, y_2, \ldots, y_d)$ such that $y_i(p) = 0$ and

$$f(y) = f(p) \pm y_1^2 \pm y_2^2 \cdots \pm y_d^2.$$

We know that the volume of level sets of quadratic functions are continuous. So, given a critical point p of f at level a, volume of level sets of f in a neighborhood U of p converge when the levels converge to a. When $x_0 \in f^{-1}(a)$ is a regular point, then there exists a neighborhood U_{x_0} such that the volume of level sets are continuous. This follows from the implicit function theorem.

Now, using compactness of $f^{-1}(a) \cap D$, we get that volume of level sets is continuous at any arbitrary level.

Since the number of critical values of f is finite in D, any critical level in D can be approximated by regular levels of f in D. Let ϵ_n be a sequence converging to zero such that $(b-\epsilon_n), (a+\epsilon_n)$ are sequences of such regular values of f. By continuity of the volume of level sets, we have

$$\mathcal{H}^{d-1}(f^{-1}(b)) - \mathcal{H}^{d-1}(f^{-1}(a)) = \lim_{n\to\infty}[L(b-\epsilon_n) - L(a+\epsilon_n)].$$

Using case 1, we have the integral formula for difference of volume of level sets. Note that

$$\left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta} \right\rangle \mathbb{1}_{f\in[a+\epsilon_n, b-\epsilon_n]} \to \left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta} \right\rangle \mathbb{1}_{f\in[a,b]},$$
$$\kappa \mathbb{1}_{f\in[a+\epsilon_n, b-\epsilon_n]} \to \kappa \mathbb{1}_{f\in[a,b]}$$

pointwise. Hence by the dominated convergence theorem, we have equation (2.1) for case 2 as well.

We have that $\mathcal{H}^{d-1}(f^{-1}(b)) \to \mathcal{H}^{d-1}(f^{-1}(a))$ as $b \to a$ a.s. by above discussion of continuity of length w.r.t levels. We also have $\mathbb{E}[\mathcal{H}^{d-1}(f^{-1}(b))] \to \mathbb{E}[\mathcal{H}^{d-1}(f^{-1}(a))]$ when $b \to a$ by Kac-Rice formula. Hence, by Scheffe's lemma, we have L^1 convergence.

□

Proof of Corollary 2.2.4. Take expectation to both sides of the equation (2.1). Switching integration and expectation because of Fubini's theorem, we get

$$\begin{aligned}\mathbb{E}[\mathcal{H}^{d-1}(f^{-1}(b))] - &\mathbb{E}[\mathcal{H}^{d-1}(f^{-1}(a))] \\ &= \iint_D \mathbb{E}[\kappa \mathbb{1}_{f\in[a,b]}] d\text{vol} - \oint_{\partial D} \mathbb{E}\left[\left\langle \frac{\nabla f}{|\nabla f|}, \hat{\eta} \right\rangle \mathbb{1}_{f\in[a,b]}\right] dS.\end{aligned}$$

Now, let us divide the above equation by $b-a$ and try taking the limit $b \to a$.

First, from stationary Kac-Rice formula, we have

$$\lim_{b\to a} \frac{\mathbb{E}[\mathcal{H}^{d-1}(f^{-1}(b))] - \mathbb{E}[\mathcal{H}^{d-1}(f^{-1}(a))]}{b-a} = -ap(a)\mathcal{L}^d(D)\mathbb{E}[|\nabla f|].$$

Next, from the continuity of the Gaussian regression formula, we get the following conditional expectations (see [AW09, Theorem 3.2] for an explanation). Consider the expression $\mathbb{E}[\kappa \mathbb{1}_{f\in[a,b]}]$ and write it in the following form,

$$\mathbb{E}[\kappa \mathbb{1}_{f\in[a,b]}] = \int_a^b \mathbb{E}[\kappa | f = u] p(u) du$$

Now note that $\mathbb{E}[\kappa|f=u]$ is continuous in u, hence $(b-a)^{-1}\mathbb{E}[\kappa \mathbb{1}_{f\in[a,b]}] \to \mathbb{E}[\kappa|f=a]$ as $b\to a$. By the dominated convergence theorem, we have

$$\lim_{b\to a}\frac{1}{b-a}\iint_D \mathbb{E}[\kappa\mathbb{1}_{f\in[a,b]}]d\mathrm{vol} = \iint_D \mathbb{E}[\kappa|f=a]p(a)d\mathrm{vol}.$$

A similar argument works for the claim

$$\lim_{b\to a}\frac{1}{b-a}\oint_{\partial D}\mathbb{E}\left[\left\langle \frac{\nabla f}{|\nabla f|},\hat{\eta}\right\rangle \mathbb{1}_{f\in[a,b]}\right]d\mathrm{S} = \oint_{\partial D}\mathbb{E}\left[\left\langle \frac{\nabla f}{|\nabla f|},\hat{\eta}\right\rangle |f=a\right]p(a)d\mathrm{S}.$$

Combining these calculations, we have the following equation

$$-ap(a)\mathcal{L}^d(D)\mathbb{E}[|\nabla f|]+\oint_{\partial D}\mathbb{E}\left[\left\langle \frac{\nabla f}{|\nabla f|},\hat{\eta}\right\rangle |f=a\right]p(a)d\mathrm{S} = \iint_D \mathbb{E}[\kappa|f=a]p(a)d\mathrm{vol} \tag{2.3}$$

where p is the pdf of standard Gaussian random variable.

Now, we claim that

$$\oint_{\partial D}\mathbb{E}\left[\left\langle \frac{\nabla f}{|\nabla f|},\hat{\eta}\right\rangle |f=a\right]d\mathrm{S}=0. \tag{2.4}$$

Since ∇f and f are pointwise independent r.v. (by stationary), integral on a $(d-1)$-dimensional slab in ∂D cancels that from the opposite slab (also by stationarity). So we have equation (2.4).

Again by stationarity of κ, the equation (2.3) reduces to $\mathbb{E}[\kappa|f=a]p(a) = -a\mathbb{E}[|\nabla f|]p(a)$. Hence we have the Corollary 2.2.4.

□

Proof of Theorem 2.2.2. First, observe that $\mathcal{H}^{d-1}(f^{-1}(a))\to 0$ almost surely as $a\to\infty$ or as $a\to-\infty$, since probability that f is unbounded on D is zero. Now, taking difference of equation (2.1) applied to f_1, f_2 and taking $b=0$, $a\to-\infty$ we have,

$$\begin{aligned}\mathcal{H}^{d-1}(f_1^{-1}(0))-\mathcal{H}^{d-1}(f_2^{-1}(0)) &= \iint_D \left[\kappa_1\mathbb{1}_{f_1\le 0}-\kappa_2\mathbb{1}_{f_2\le 0}\right]d\mathrm{vol}\\ &\quad -\int_{\partial D}\left[\left\langle \frac{\nabla f_1}{|\nabla f_1|},\hat{\eta}\right\rangle \mathbb{1}_{f_1\le 0}-\left\langle \frac{\nabla f_2}{|\nabla f_2|},\hat{\eta}\right\rangle \mathbb{1}_{f_2\le 0}\right]dS.\end{aligned} \tag{2.5}$$

We bound the bulk term and the boundary term of equation (2.5) separately.

Bulk term: First we have,

$$\left|\int_D [\kappa_1 \mathbb{1}_{f_1\le 0} - \kappa_2 \mathbb{1}_{f_2\le 0}]\, d\mathrm{vol}\right| \le \left|\int_D (\kappa_1-\kappa_2)\mathbb{1}[f_1,f_2<0] d\mathrm{vol}\right| + \left|\int_D \kappa_1 \mathbb{1}[f_1f_2<0] d\mathrm{vol}\right| + \left|\int_D \kappa_2 \mathbb{1}[f_1f_2<0] d\mathrm{vol}\right|. \tag{2.6}$$

For the second term of equation (2.6) we show that, with probability close to one, $\mathcal{L}^d(f_1f_2<0)$ is small and that integral of curvature is bounded with high probability.

Note that $\mathbb{E}[|\kappa_1|^{1+\alpha}]<\infty$ for all $0<\alpha<1$ (see section 2.3.1). Take $\alpha=1/2$ when applying Hölder inequality in the following computation. Given a point $x\in D$, recall that $\kappa_1(x)$ is the mean curvature of the level set $f^{-1}(c)$, where $x\in f^{-1}(c)$, at x.

$$\begin{aligned}\left|\mathbb{E}\int_D \kappa_1 \mathbb{1}[f_1f_2<0] d\mathrm{vol}\right| &\le \mathbb{E}\left|\int_D \kappa_1 \mathbb{1}[f_1f_2<0] d\mathrm{vol}\right| \\ &\le \int_D \mathbb{E}|\kappa_1 \mathbb{1}[f_1f_2<0]| d\mathrm{vol} \\ &\le (\mathbb{E}|\kappa_1|^{3/2})^{2/3} \int_D \mathbb{P}[f_1(x)f_2(x)<0]^{1/3} d\mathrm{vol} \\ &\le C_1\cdot \mathcal{L}^d(D)\sup_D[(\arccos(\rho(x)))^{1/3}]\end{aligned} \tag{2.7}$$

where $\rho(x)$ is the correlation between $f_1(x)$ and $f_2(x)$,and the constant C_1 depends only on the law of the fields. Note that $\arccos(x)=c_1\sqrt{(1-x)}+O((1-x)^{3/2})$ near $x=1$, where c_1 is a universal constant. We have that $|1-\rho(x)|\le \sigma_D^2/2$ for all $x\in D$. Hence we have,

$$\mathbb{E}\left[\left|\int_D \kappa_1 \mathbb{1}[f_1f_2<0] d\mathrm{vol}\right|\right] \le C_2\mathcal{L}^d(D)\sigma_D^{1/3} \tag{2.8}$$

where the constant C_2 only depends on the spectral measure.

Next, we'll bound the term

$$\mathbb{E}\left[\left|\int_D (\kappa_1-\kappa_2)\mathbb{1}[f_1,f_2<0] d\mathrm{vol}\right|\right].$$

Notice that

$$\mathbb{E}\left[\left|\int_D (\kappa_1-\kappa_2)\mathbb{1}[f_1,f_2<0] d\mathrm{vol}\right|\right] \le \mathbb{E}\left[\int_D |\kappa_1-\kappa_2| d\mathrm{vol}\right]$$

We split the computation into two cases: $||\nabla f_i|| < \delta$ for one of the $i = 1, 2$ and $||\nabla f_i|| > \delta$ for both i's (for some fixed $\delta > 0$). 2 Now,

$$\begin{aligned}\int_D \mathbb{E}\left[|\kappa_1 - \kappa_2| \mathbb{1}[||\nabla f_1|| < \delta]\right] d\mathrm{vol} &\leq (\mathbb{E}|\kappa_1 - \kappa_2|^{4/3})^{3/4} \int_D \mathbb{P}(||\nabla f_1||^2 < \delta^2)^{1/4} d\mathrm{vol} \\ &\leq C_3 \mathcal{L}^d(D)\sqrt{\delta}.\end{aligned} \tag{2.9}$$

In the first inequality, we used the fact that curvature has $1 + \alpha$ moments for $\alpha \in ([0, 1))$ and applied Hölder's inequality. Observe that $||\nabla f_1||^2$ has bounded pdf around zero, so $\mathbb{P}(||\nabla f_1||^2 < \delta^2) = O(\delta^2)$.
Define

$$\beta := ||f_1 - f_2||_{C^2(D)}.$$

We exploit explicit representation of the curvature (2.15) in terms of derivatives of the field. Given that $||\nabla f_1||, ||\nabla f_2|| > \delta$ we have,

$$|\kappa_1 - \kappa_2| \leq \frac{1}{\delta^3}(\beta p_1 + \beta^2 p_2 + \beta^3 p_3)$$

where p_i's are polynomials in the first two derivatives of f_1 of degree at most 2. Hence,

$$\mathbb{E}\left|\int_D (\kappa_1 - \kappa_2)\mathbb{1}[||\nabla f_1||, ||\nabla f_2|| > \delta] d\mathrm{vol}\right| \leq \delta^{-3} \int_D \mathbb{E}[(\beta p_1 + \beta^2 p_2 + \beta^3 p_3)] d\mathrm{vol}. \tag{2.10}$$

Using Cauchy-Schwartz inequality and the fact that laws of the polynomials p_is are translation invariant, we have the following estimate,

$$\mathbb{E}\left|\int_D (\kappa_1 - \kappa_2)\mathbb{1}[||\nabla f_1||, ||\nabla f_2|| > \delta] d\mathrm{vol}\right| \leq \frac{C_4 \mathcal{L}^d(D)}{\delta^3}(\sqrt{\mathbb{E}\beta^2} + \sqrt{\mathbb{E}\beta^4} + \sqrt{\mathbb{E}\beta^6}).$$

But we have the moment estimates of β,

$$\mathbb{E}[\beta^p] \leq \tilde{C}\sigma_D^p$$

which is given in [NS16, A.11.1], the $\mathbb{E}\beta^2$ term dominates when the coupling of the fields f_1, f_2 close. So we have,

$$\mathbb{E}\left|\int_D (\kappa_1 - \kappa_2)\mathbb{1}[||\nabla f_1||, ||\nabla f_2|| > \delta] d\mathrm{vol}\right| \leq \frac{C_5 \mathcal{L}^d(D)}{\delta^3}(\sqrt{\mathbb{E}\beta^2}). \tag{2.11}$$

Boundary term: We come to the boundary term of equation (2.5).

$$\begin{aligned}&\int_{\partial D} \left[\left\langle \frac{\nabla f_1}{|\nabla f_1|}, \hat{\eta}\right\rangle \mathbb{1}_{f_1 \leq 0} - \left\langle \frac{\nabla f_2}{|\nabla f_2|}, \hat{\eta}\right\rangle \mathbb{1}_{f_2 \leq 0}\right] dS = \\ &\int_{\partial D} \left[\left\langle \frac{\nabla f_1}{|\nabla f_1|} - \frac{\nabla f_2}{|\nabla f_2|}, \hat{\eta}\right\rangle \mathbb{1}[f_1, f_2 < 0]\right] dS + \int_{\partial D} \left[\left\langle \frac{\nabla f_1}{|\nabla f_1|}, \hat{\eta}\right\rangle \mathbb{1}[f_1 < 0, f_2 > 0]\right] dS \\ &\qquad + \int_{\partial D} \left[\left\langle \frac{\nabla f_2}{|\nabla f_2|}, \hat{\eta}\right\rangle \mathbb{1}[f_2 < 0, f_1 > 0]\right] dS\end{aligned} \tag{2.12}$$

The analysis of bounds of first term of RHS of equation (2.12) is similar to that of equation (2.9). We get that,

$$\left| \mathbb{E} \int_{\partial D} \left[\left\langle \frac{\nabla f_1}{|\nabla f_1|} - \frac{\nabla f_2}{|\nabla f_2|}, \hat{\eta} \right\rangle \mathbb{1}[f_1, f_2 < 0] \right] dS \right| \leq C_6 \mathcal{L}^{d-1}(\partial D)(\delta_1^2 + \mathbb{E}\beta/\delta_1) \tag{2.13}$$

for $\delta_1 > 0$.
Now, second term of RHS is bounded by $C \cdot \mathcal{L}^{d-1}(\partial D \cap \{f_1 f_2 < 0\})$ since $\nabla f_1 / |\nabla f_1|$ is unit vector. By similar argument which lead to equation (2.8), we have

$$\mathbb{E}\mathcal{L}^{d-1}(\partial D \cap \{f_1 f_2 < 0\}) \leq C_8 \mathcal{L}^{d-1}(\partial D)\sigma_D. \tag{2.14}$$

This is again dominated by the quantity of RHS of equation (2.8).

Analysis of the final bound: We combine the bounds from (2.8), (2.9),(2.11), and (2.13). Finally, we get

$$\mathbb{E}|\mathcal{H}^{d-1}(f_1^{-1}(0)) - \mathcal{H}^{d-1}(f_2^{-1}(0))| \leq C\mathcal{L}^d(D) \left(\sigma_D^{1/3} + \sqrt{\delta} + \frac{\sqrt{\mathbb{E}\beta^2}}{\delta^3} + \frac{\delta_1^2}{R} + \frac{\mathbb{E}\beta}{\delta_1 R} \right).$$

Estimates from [NS16, A.9, A.11.1] gives us $\mathbb{E}\beta \leq C_1(R)\sigma_D$ and $\sqrt{\mathbb{E}\beta^2} \leq C_2(R)\sigma_D$, where we can show that $C_1(R), C_2(R)$ behave like $\sqrt{\log R}$. One way to argue is to cover the domain D with discs of fixed radius and proceed as in [BM22, Lemma 2.1].

Now, choosing $\delta = \sigma_D^{2/7}, \delta_1 = \sigma_D^{1/2}$, and assuming σ_D is small enough we have,

$$\mathbb{E}|\mathcal{H}^{d-1}(f_1^{-1}(0)) - \mathcal{H}^{d-1}(f_2^{-1}(0))| \leq C(R)\mathcal{L}^d(D)\sigma_D^{1/7}.$$

□

2.3.1 Technical bits

Moments of curvature r.v.: We show that the $(1+\alpha)$-moments are finite, where $0 < \alpha < 1$ for the r.v. κ of a C^2-smooth, non-degenerate, stationary field f. Observe that

$$\kappa = \frac{|\nabla f|^2 \mathrm{Tr}(H(f)) - \nabla f H(f) \nabla f^{\mathrm{T}}}{|\nabla f|^3} \tag{2.15}$$

where $H(f)$ is the Hessian of the function f, by a simple algebraic computation.

First let us prove that $\mathbb{E}[|\kappa|^{1+\alpha}] < \infty$ for $d = 2$ case. The general case follows from similar computation. Observe that

$$\mathbf{X} = (x_1, x_2, x_3, x_4, x_5) = (\partial_x f, \partial_y f, \partial_{xx} f, \partial_{xy} f, \partial_{yy} f)$$

is a Gaussian vector and that

$$(\partial_x f, \partial_y f) \text{ and } (\partial_{xx} f, \partial_{xy} f, \partial_{yy} f)$$

are independent, by stationarity of the field f. Let Σ be the covariance matrix of the Gaussian vector $(\partial_x f, \partial_y f)$ and $\mathbb{P}_1$ be the law of $(\partial_{xx} f, \partial_{xy} f, \partial_{yy} f)$. Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}' = (x_3, x_4, x_5)$.

So,

$$\begin{aligned}\mathbb{E}[|\kappa|^{1+\alpha}] =& \frac{1}{\sqrt{\det(2\pi\Sigma)}} \times \\ & \int_{\mathbb{R}^5} \left| \frac{x_2^2 x_3 - 2x_1 x_2 x_4 + x_1^2 x_5}{(x_1^2 + x_2^2)^{3/2}} \right|^{1+\alpha} \exp\left(-1/2(\mathbf{x}^T \Sigma^{-1} \mathbf{x})\right) d\mathbf{x} d\mathbb{P}_1(\mathbf{x}').\end{aligned}$$

By changing the variables to $x_1 = r\cos\theta, x_2 = r\sin\theta$ and keeping other variables same, we get ,

$$\begin{aligned}\mathbb{E}[|\kappa|^{1+\alpha}] =& \frac{1}{\sqrt{\det(2\pi\Sigma)}} \times \\ & \int_I r^{-\alpha} \left|\sin^2\theta x_3 - \sin(2\theta) x_4 + \cos^2\theta x_5\right|^{1+\alpha} \exp(-1/2(\tilde{\mathbf{x}}^T \Sigma^{-1} \tilde{\mathbf{x}})) dr d\theta d\mathbb{P}_1(\mathbf{x}')\end{aligned}$$

where $\tilde{\mathbf{x}} = (r\cos\theta, r\sin\theta)$ and $I = [0, \infty] \times [0, 2\pi] \times \mathbb{R}^3$. Now, for $0 \leq \alpha < 1$ the above integral converges. Near the origin of I convergence is taken care by $\int_0^1 r^{-\alpha} dr < \infty$ and away from origin $\exp(\cdots)$ dominates. The result follows from the fact that the vector $(\partial_{xx} f, \partial_{xy} f, \partial_{yy} f)$ has all moments finite.

Integrability of curvature function: Consider a deterministic C^2-Morse function f on a compact domain $D \subset \mathbb{R}^d$. As above, at every $x \in D$ which is a regular point of f, define κ to be the divergence of unit normal of f.
We prove that

$$\int_D |\kappa| d\text{vol} < \infty.$$

Note that except at critical points of f, κ is continuous. So just need to show that $\int_{B_r(x_0)} |\kappa| d\text{vol} < \infty$ for a critical point x_0 of f and a small enough ball $B_r(x_0)$ around x_0.

We have $\nabla f(x) = H(f)|_{x_0}(x - x_0) + O(||x - x_0||^2)$, by Taylor's series. Since f is Morse, we can invert $H(f)|_{x_0}$ to have

$$||\nabla f(x)|| \geq C \frac{||x - x_0||}{||H(f)_{x_0}^{-1}||}.$$

Since $\partial_{xx} f, \partial_{xy} f, \partial_{yy} f$ are all bounded on D and

$$|\partial_x f(x)| \leq c_1 ||x - x_0||, |\partial_y f(x)| \leq c_2 ||x - x_0||$$

near x_0 and again, exploiting the equation (2.15), we have

$$\int_{B_r(x_0)} |\kappa| d\text{vol} \leq \tilde{C} \int_{B_r(x_0)} \frac{1}{||x - x_0||} d\text{vol}.$$

But we have

$$\int_{B_r(x_0)} \frac{1}{||x - x_0||} d\text{vol} < \infty$$

for any $d \geq 2$. This completes the proof that the mean curvature function is integrable on D.

Chapter 3

Structure of critical points of smooth Gaussian fields

3.1 Introduction

Local maxima / high points of Gaussian fields is an important geometric observable in probability theory, mathematical physics and in natural sciences. Analysis of critical points of smooth fields is crucial in the understanding of of landscape of the field. For example, it plays an important role in computing topological quantities like number of connected components of level sets [BMM22].

In statistics, extreme values of Gaussian processes are vital to real-world applications and are studied well. Limit theorems for extrema of these processes were proved in 1960's & 70's cf. [LLR83]. Then later in 1990's, Piterbarg [Pit96] showed Poisson process convergence for so-called 'A-exit points' over a high level of a smooth Gaussian field of dimension $d \geq 2$.

The following are some of the results pertaining to Poisson convergence of point processes of smooth Gaussian fields (including dimension one).

1. In 1-dim, number of upcrossings at level $u(T) \simeq \sqrt{2 \log T}$ over the interval $[0, T]$, as $T \to \infty$. [LLR83, Chapter 9]

2. In dimension 2 or more, "A-exit points" over level $u(R) \simeq \sqrt{2d \log R}$ in growing region $[0, R]^d$, as $R \to \infty$. [Pit96, Section 15]

3. In dimension 2 or more, for local maxima over level $u(R) \simeq \sqrt{2d \log R}$ in growing region $[0, R]^d$, as $R \to \infty$. [Qi22, Chapter 3]

In all of the examples above, the decay of correlation of the field at infinity is around $\log^{-1}$ of the distance. A somewhat related set of results include limit theorems for extremal processes for class processes with Markovian property like Gaussian free field, branching Brownian motion. Arguin et al. [ABK13] showed that extremal process of branching Brownian motion converges weakly to clustered Poisson process. Oleskar-Taylor, Sousi [ST20] showed that high points (level above $\alpha\mathbb{E}[\text{maxima}], 0 < \alpha_0 < \alpha$) of discrete GFF in $d \geq 3$ converges in total variation distance to independent Bernoulli process on the lattice. In essence, we can expect some Poisson limit for extremal process if either the covariance decays fast enough at infinity or there's some Markov property.

Our contribution is to consider limits for local maxima over arbitrary levels $u(R) \to \infty$ as $R \to \infty$. As far as we know, this is the first time a lower level than the expected maxima in a domain is studied in this context. Also, our result includes monochromatic random waves (MRW) model, which is not covered in [Qi22]. Surprisingly, lowering the rate of threshold level does not impose any additional condition on the decay rate of correlations.

Let us remark about the landscape of random planes waves in the context of Thm 3.2.2. Numerical simulations by A. Barnett (See Figure 5.1) suggests that there's apparent filament structure of extrema above a level (say above three std deviations of the field). Our result indicates that these patterns disappear at high levels.

3.2 Setup and statement

Consider a C^{2+}-smooth Gaussian field $f : \mathbb{R}^d \to \mathbb{R}$, with $d \geq 2$, $\mathbb{P}$ being the associated probability measure. Let $\mathbb{E}$ denote the expectation with respect to $\mathbb{P}$ and let $r(x, y) = \mathbb{E}[f(x)f(y)]$ be the covariance kernel [see [NS16, Appendix A] for more details]. For $R > 0$, $L = [0, 1]^d \subset \mathbb{R}^d$ and let $L_R = [0, R]^d = R \cdot L$. Let $u : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function such that $u(R) \leq \sqrt{2d \log R} \to \infty$ as $R \to \infty$. By abuse of notation, we often write $u = u(R)$.

Consider the following scaling of the field f,

$$f_R(x) = f(\mu(R) \cdot x) \quad \text{for } x \in \mathbb{R}^d,$$

where

$$\mu(R)^{-1} = \kappa^{1/d} R u^{\frac{d-1}{d}} \exp\left(-\frac{u^2}{2d}\right) \text{ and } \kappa = 1/(2\pi)^{(d+1)/2}.$$

Now we define a sequence of point process indexed by R as follows. Let

$$\eta_R(B) = \text{ number of local maxima above level } u(R) \text{ of the field } f_R \text{ in } B$$

where B is a Borel set in $\mathbb{R}^d$. Let

$$\Phi_R(B) = \eta_R(R \cdot B)$$

for Borel sets B in $\mathbb{R}^d$. Our goal is to show that $\Phi_R \to \Phi$ *weakly* as point processes where Φ is a homogeneous Poisson point process, given that the field

f satisfies some mild regularity and correlation decay conditions (see [Kal17, Chapter 4]).

Assumptions 3.2.1. *Throughout the article, we impose the following conditions on the Gaussian field f.*

1. *Centred ($\mathbb{E}[f(x)] = 0$), stationary ($r(x,y) = r(x-y)$), normalised ($\mathbb{E}[f(x)^2] = 1$) for all $x, y \in \mathbb{R}^d$.*
2. *Decay of correlation: $r(x) = o((\log \|x\|)^{1-d})$ as $x \to \infty$.*
3. *The vector $(f(0), \nabla f(0))$ has density in $\mathbb{R}^{d+1}$. In addition, either the vector $(f(0), \nabla f(0), \nabla^2 f(0))$ has density in $\mathbb{R}^{(d+1)+d(d+1)/2}$ or f is isotropic field (i.e. $r(x) =$ "$r(\|x\|)$").*
4. *Local structure: $r(x) = 1 - \|x\|^2 + o(\|x\|^2)$ as $x \to 0$. Note that $\exists$ invertible matrix M such that $r(M \cdot x) = 1 - \|x\|^2 + o(\|x\|^2)$ as $x \to 0$ for any C^3-smooth field f.*

One observation regarding the covariance structure r is that

$$r(x,y) < 1 \quad \forall x \neq y.$$

This follows from stationarity of the field and the fact that $r(x) \to 0$ as $x \to \infty$. This is helpful when estimating exceedance probability of the field over a large given threshold.

Theorem 3.2.2. *With the setup above and with the Assumptions 3.2.1 on the Gaussian field $f : \mathbb{R}^d \to \mathbb{R}$, we have*

$$\Phi_R \to \Phi \quad \textit{in distribution as } R \to \infty$$

where Φ is Poisson point process with intensity measure as Lebesgue measure on $\mathbb{R}^d$.

First, note that invertible linear transform T of a Poisson point process (with intensity measure λ) is again a Poisson point process with new intensity measure $|\det(T)|\lambda$. So rescaling the field as in 4. of the assumption above is just for convenience. Next, Bargmann-Fock field and monochromatic random waves for dimension $d \geq 2$ satisfy the assumptions. Indeed, the covariance kernels have decay rates $\exp(-\|x\|^2/2)$ and $O(\|x\|^{-1/2})$ for Bargmann-Fock and monochromatic random waves respectively.

Now, some comments on the scaling of the field. By Appendix A, expected maximum of the field in the region $[0,R]^d$ is asymptotically $\sqrt{2d \log R}$ as $R \to \infty$. Now by super-concentration of maximum for smooth Gaussian field result [Tan15], variance of maxima behaves like $1/\log R$. Hence, for levels above $\alpha\sqrt{2d \log R}$ with $\alpha > 1$, we don't expect to see any point in $[0,R]^d$. So we assume $u \leq \sqrt{2d \log R}$ (supercritical case of $\alpha > 1$ is taken care in 'Chen-Stein method' section anyway).

Let us illustrate our scaling procedure by taking the level to be $u = \sqrt{2d\alpha \log R}$. To compare it to a homogeneous Poisson process, we need to rescale the local maxima point process to, say, unit density. Let $M_u(f, S)$ denote the number of local maxima of f in $S \subset \mathbb{R}^d$ with $f > u$. Then,

$$\mathbb{E}[M_u(f, [0, R]^d)] \simeq (\log R)^{(d-1)/2} R^{(1-\alpha)d}.$$

Rescaling the point process in $[0, R]^d$ by factor $R^{-\alpha}$ (ignoring log factors), we get a unit density process, which corresponds to Φ. Note that we've defined Φ above by reversing this procedure.

Plan of proof

It is well known at least since 1970's that avoidance probabilities (i.e. $\mathbb{P}(\eta(B) = 0)$ for Borel sets B) characterise simple point process (i.e. point processes with mass concentrated only on atoms). Now, weak convergence of these point processes can be studied by scrutinising avoidance probabilities and intensity measures.

Definition 3.2.3 (DC-ring). *Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}^d$. A ring $\mathcal{L} \subset \mathcal{B}$ is called a DC-ring ('dissecting covering' ring) if for any compact set K from $\mathcal{B}$, and arbitrary $\epsilon > 0$, there exists a finite covering of K by some sets $l \in \mathcal{L}$ such that diam $L \leq \epsilon$.*

Let $\mathcal{L}$ be a ring generated by rectangles

$$\prod_{i=1}^{d} [t_i, t_i + s_i), \quad s_i \geq 0, i = 1, 2, \ldots, d$$

which will be a DC-ring with the property that $\Phi(\partial l) = 0$ a.s. for any $l \in \mathcal{L}$. Then by [Kal17, Theorem 4.18], it is enough to show that

$$\lim_{R\to\infty} \mathbb{P}(\Phi_R(l) = 0) = \mathbb{P}(\Phi(l) = 0), \quad \limsup_{R\to\infty} \mathbb{E}\Phi_R(l) \leq \mathbb{E}\Phi(l) \tag{3.1}$$

for all $l \in \mathcal{L}$. In the proof below, we'll show this for $L = [0, 1]^d$ but the argument works for any $l \in \mathcal{L}$.

First, we approximate avoidance probabilities of the sequence Φ_R by the excursion probabilities of the field f_R (Lemma 3.3.1). Then we approximate excursion probabilities on rectangles $\mathbb{P}(\sup_{R \cdot L} f_R > u)$ by that on a grid which is fine enough (Lemma 3.3.2). By standard theory, we know that for a regular enough field f with unit variance, the excursion set $\{f > u\}$ is captured by a grid with width u^{-1} for large u. Now we compare the excursion probabilities of the field f to that of the field f_0 which is an i.i.d copy of f on a each fixed box. This is done by comparison method for Gaussian vectors [Pit96, Thm 1.1] and is the same as proof of [Pit96, Thm15.2]. Lastly, from Lemma 3.3.4 we show that excursion probabilities of the field f_0 converges to avoidance probabilities of Poisson point process, which proves the first part of eq. (3.1).

We consider the second part of eq. (3.1). Computing expected number of critical points of given index of smooth Gaussian fields is classical problem in this field [Adl10]. Thanks to Kac-Rice formulas, we know precise estimates of these quantities, even explicit result in some cases. Using these estimates, we'll show that

$$\lim_{R\to\infty} \mathbb{E}[\Phi_R(L)] = \mathbb{E}[\Phi(L)].$$

These two parts conclude the proof of the theorem 3.2.2.

3.3 Proof

Recall that L is a unit box in $\mathbb{R}^d$ and let $L_R := R \cdot L$. Define

$$P_f(u,S) = \mathbb{P}\left(\sup_{t\in S} f(t) \le u\right) \qquad \text{and} \qquad \bar{P}_f(u,S) = \mathbb{P}\left(\sup_{t\in S} f(t) \ge u\right).$$

Let A be a ball centred at origin in $\mathbb{R}^d$. We define Minkowski sum of two subsets A, B of $\mathbb{R}^d$ as

$$A \oplus B = \{x + y : x \in A, y \in B\}.$$

Now we approximate the avoidance probability of point process with excursion probabilities.

Lemma 3.3.1. *With the above setup, we have*

$$\mathbb{P}(\Phi_R(L) = 0) = P_{f_R}(u, L_R) + o(1) \qquad \text{as } R \to \infty.$$

Proof. First, observe that $\mathbb{P}(\Phi_R(L) = 0) \ge P_{f_R}(u, L_R)$. From the fact that each connected component of $\{f(x) \ge u\}$ must have a local maximum, we have

$$\{\Phi_R(L) > 0\} \supseteq \left\{\sup_{L_R} f_R \ge u, \sup_{(L_R\oplus A)\setminus L_R} f_R < u\right\}.$$

Note that the RHS just makes sure that L_R has at least one component of $\{f_R(x) \ge u\}$ lying completely inside it. Hence,

$$\mathbb{P}(\Phi_R(L) = 0) \le P_{f_R}(u, L_R) + \mathbb{P}\left(\sup_{L_R} f_R \ge u, \sup_{(L_R\oplus A)\setminus L_R} f_R \ge u\right).$$

Now,

$$\mathbb{P}\left(\sup_{L_R} f_R \geq u, \sup_{(L_R \oplus A)\setminus L_R} f_R \geq u\right) \leq \bar{P}_{f_R}(u, (L_R \oplus A) \setminus L_R).$$

Noting that $\mathrm{vol}((L_R \oplus A) \setminus L_R) = O(R^{d-1})$ for large R and that $\bar{P}_{f_R}(u, S) = \bar{P}_f(u, \mu(R) \cdot S)$ and applying [Pit96, Thm 7.1], using homogeneity of the field, we have

$$\begin{aligned}\bar{P}_{f_R}(u, (L_R \oplus A) \setminus L_R) \leq & C \cdot \mathrm{vol}(\mu(R) \cdot (L_R \oplus A) \setminus L_R) u^{d-1} \exp(-u^2/2) \\ = & O(R^{-1}) \quad \text{as } R \to \infty.\end{aligned}$$

□

Now, we discretise the domain and approximate the excursion probabilities on this grid as explained before. Let g_R be some scaling (to be determined in the course of the proof). Fixing $b > 0$, define $\mathcal{R}_b = b g_R \mathbb{Z}^d$.

Lemma 3.3.2. *For any $\epsilon > 0$, there exists $b, R_0 > 0$ such that for all $R > R_0$,*

$$P_{f_R}(u, L_R \cap \mathcal{R}_b) - P_{f_R}(u, L_R) \leq \epsilon.$$

Proof. We have

$$P_{f_R}(u, L_R \cap \mathcal{R}_b) - P_{f_R}(u, L_R) = \mathbb{P}\left(\sup_{L_R \cap \mathcal{R}_b} f_R \leq u, \sup_{L_R} f_R > u\right).$$

By homogeneity of the field f_R, we have (calling $\mu(R)\mathcal{R}_b = \mathcal{R}'_b$)

$$\mathbb{P}\left(\sup_{L_R \cap \mathcal{R}_b} f_R \leq u, \sup_{L_R} f_R > u\right) \leq (R\mu(R))^d \mathbb{P}\left(\sup_{L \cap \mathcal{R}'_b} f \leq u, \sup_{L} f > u\right)$$

Now by the standard theory of excursion approximation (see [Pit96, Lemma 15.3]), when $g_R = (u\mu(R))^{-1}$ and $b > 0$ is small enough, we have

$$\mathbb{P}\left(\sup_{[0,1]^d \cap \mathcal{R}'_b} f \leq u, \sup_{[0,1]^d} f > u\right) \leq \epsilon \quad , R > R_0.$$

□

We define $\lambda_{a,R}$ given numbers $a > \delta > 0$. Divide the rectangle $\mu(R)R \cdot L$ into smaller ones by following construction. Divide each edge of $\mu(R)R \cdot L$ into segments of length 'a' alternated by that of δ. Call $\lambda_{a,R}$ the union of cubes of side length a. Note that the distance between the cubes are greater than δ. The following lemma says that if gap between the cubes of $\lambda_{a,R}$ are small enough, then the excursion probabilities are close to that the discretisation of $R\mu(R) \cdot L$.

Lemma 3.3.3. *For any $a, \epsilon > 0$ given, there exists $\delta > 0$, such that, for all R large enough we have,*

$$P_f(u, \lambda_{a,R} \cap \mathcal{R}'_b) - P_f(u, \mu(R)R \cdot L \cap \mathcal{R}'_b) \leq \epsilon.$$

Proof. We have that

$$P_f(u, \lambda_{a,R} \cap \mathcal{R}'_b) - P_f(u, \mu(R)R \cdot L \cap \mathcal{R}'_b) \leq \mathbb{P}\left(\sup_{\lambda_{a,R} \cap \mathcal{R}'_b} f \leq u, \sup_{\mu(R)R \cdot L \cap \mathcal{R}'_b} f > u \right).$$

Now using homogeneity of the field,

$$\begin{aligned}
\mathbb{P}\left(\sup_{\lambda_{a,R} \cap \mathcal{R}'_b} f \leq u, \sup_{\mu(R)R \cdot L \cap \mathcal{R}'_b} f > u \right) &\leq \bar{P}_f(u, \mu(R)R \cdot L \setminus \lambda_{a,R}) \\
&\leq \mathrm{vol}(\mu(R)R \cdot L \setminus \lambda_{a,R}) \bar{P}_f(u, L) \\
&\leq \delta \frac{(\mu(R)R)^d}{(a+\delta)} \bar{P}_f(u, L) \\
&\leq C\delta((\mu(R)R)^d) u^{d-1} \exp(-u^2/2)
\end{aligned}$$

Now, we get that the expression is bounded by $c \cdot \delta$ where c is a constant which doesn't depend on R. □

Let f_0 be a field defined on $\lambda_{a,R}$ such that on the cubes of side length a, the field is made up of i.i.d copies of f. We now show that the excursion probability of f_0 converges to avoidance probability of Poisson point process.

Lemma 3.3.4. *We have*

$$P_{f_0}(u, \lambda_{a,R}) \to \exp(-vol(L)) \quad \textit{as } R \to \infty.$$

Proof. Let N be the number of cubes of side length a in $\lambda_{a,R}$. Then,

$$P_{f_0}(u, \lambda_{a,R}) = (1 - \bar{P}_f(u, [0,a]^d))^N$$

by independence of the field on these cubes. Taking logarithm, it is enough to estimate

$$N \log(1 - \bar{P}_f(u, [0,a]^d)) = -N\bar{P}_f(u, [0,a]^d) + O(N\bar{P}_f(u, [0,a]^d)^2).$$

Now,

$$\bar{P}_f(u, [0,a]^d) = \kappa a^d u^{d-1} \exp(-u^2/2)(1 + o(1))$$

and

$$N = \left(\frac{R\mu(R)}{a+\delta} \right)^d + O((R\mu(R))^{d-1}).$$

Hence,

$$N\bar{P}_f(u, [0,a]^d) = \left(\frac{a}{a+\delta} \right)^d + o(1) \text{ and } N\bar{P}_f(u, [0,a]^d)^2 = o(1).$$

Since L is a unit box and we can take δ arbitrarily small, we have the result. □

Proof of Theorem 3.2.2. First, observe that all the proof of lemmas goes through even when L is a finite union of finite rectangles. For any given $\epsilon > 0$, there exists a, b, δ, R_0 such that for all $R > R_0$,

$$|\mathbb{P}(\Phi_R(L) = 0) - P_f(u, \lambda_{a,R} \cap \mathcal{R}'_b)| \leq \epsilon.$$

We show that $|P_f(u, \lambda_{a,R} \cap \mathcal{R}'_b) - P_{f_0}(u, \lambda_{a,R} \cap \mathcal{R}'_b)| \to 0$ as $R \to \infty$ then by Kallenberg's theorem (see [Pit96, Section 13]) we're done. This is done by method of comparison for Gaussian vectors as in Theorem 1.1 of Piterbarg, which is a generalisation of the classical Berman inequality. For the rest of the proof, we follow argument of proof of Thm 15.2 of [Pit96].

Let K_i be a renumbering of cubes with edges of length a which comprise $\lambda_{a,R}$, $i = 1, 2, \ldots, N$. Let covariance of the field f_0 on $\lambda_{a,R}$ be denoted by $r_0(t, s)$. Define $\lambda'_{a,R,b} = \lambda_{a,R} \cap \mathcal{R}'_b$ Then by Theorem 1.1 of [Pit96], we have

$$\begin{aligned} |P_f(u, \lambda'_{a,R,b}) - P_{f_0}(u, \lambda'_{a,R,b})| \leq \frac{1}{\pi} \sum_{t,s \in \lambda'_{a,R,b}} |r(t-s) - r_0(t-s)| \\ \times \int_0^1 (1 - (hr(t,s))^2)^{-1/2} \exp\left(-\frac{u^2}{1 + hr(t,s)}\right) dh. \end{aligned} \tag{3.2}$$

Denote the summand on RHS of above equation by $\beta(t, s)$. If $t, s \in K_i$ for some i, then $r(t, s) = r_0(t, s)$, hence $\beta(t, s) = 0$.
Now consider the case that t, s belong to different K_i and K_j such that $|t-s| \leq R^{\gamma_1}$, where $\gamma_1 > 0$ is a constant chosen later. Since t, s belong different cubes, we have $|t - s| > \delta$, hence $|1 - r(t, s)| > \gamma_2 > 0$. So,

$$\frac{1}{1 + r(t, s)} > \frac{1}{2} + \frac{\gamma_2}{4}.$$

Now,

$$\begin{aligned} \sum_{\substack{t \in K_i, s \in K_j, i \neq j, \\ |t-s| < R^{\gamma_1}}} \beta(t, s) &\leq C_1 \sum |r(t, s)| \exp\left(-\frac{u^2}{1 + r(t, s)}\right) \\ &\leq C_2 (\mu(R)R)^d R^{\gamma_1 d} \exp(-u^2/2(1 + \gamma_2/2)) \\ &\leq C_3 u^{d-1} e^{u^2/2} e^{\gamma_1 u^2/2} \exp(-u^2/2(1 + \gamma_2/2)) \\ &\to 0 \quad \text{as} \quad R \to \infty \quad \text{if} \quad 0 < \gamma_1 < \gamma_2. \end{aligned} \tag{3.3}$$

Here we've used that $u \leq \sqrt{2d \log R}$, and C_i's are different constants not depending on R.

Lastly, we consider the case where $|t-s| \geq R^{\gamma_1}$. We have,

$$\begin{aligned}
\sum_{\substack{t\in K_i, s\in K_j, i\neq j,\\ |t-s|\geq R^{\gamma_1}}} \beta(t,s) &\leq C_1 \sum |r(t,s)| \exp\left(-\frac{u^2}{1+r(t,s)}\right) \\
&\leq C_2(\mu(R)R)^{2d} r'(R^{\gamma_1}) \exp\left(-\frac{u^2}{1+r'(R^{\gamma_1})}\right) \\
&\leq C_3 u^{2d-2} r'(R^{\gamma_1}) \exp\left(\frac{r'(R^{\gamma_1})u^2}{1+r'(R^{\gamma_1})}\right)
\end{aligned} \tag{3.4}$$

where

$$r'(h) := \max_{|t|\geq h} |r(t,0)|, \quad h \in (0,\infty).$$

Observing that the assumption on the decay of correlation (point 2 of Assumption 3.2.1) implies that $u^{2d-2}r'(R^{\gamma_1}) \to 0$, since we have $u \leq \sqrt{2d\log R}$ and $d \geq 2$. In particular, $u^2 r'(R^{\gamma_1}) \to 0$. Hence,

$$\sum_{\substack{t\in K_i, s\in K_j, i\neq j,\\ |t-s|\geq R^{\gamma_1}}} \beta(t,s) \to 0 \quad \text{as} \quad R \to \infty.$$

□

Computation of expectation

Our next goal is to show the following

$$\lim_{R\to\infty} \mathbb{E}[\Phi_R(L)] = \mathbb{E}[\Phi(L)].$$

For the case that $(f(0), \nabla f(0), \nabla^2 f(0))$ having density in $\mathbb{R}^{(d+1)+d(d+1)/2}$, [Adl10, Thm 6.3.1] suffices. If the field is isotropic and if $(f(0), \nabla f(0), \nabla^2 f(0))$ is degenerate then the field has to be monochromatic random wave (MRW) (see [CS18, Prop 3.10]). From Example 3.15 of [CS18], we can calculate the limit of $\mathbb{E}[\Phi_R(L)]$ for the case $d = 2$. But explicit expressions for height densities are hard to get for $d \geq 3$ directly. So we shift the MRW field by an independent normal random variable, so that the joint vector of the field, its gradient, and hessian has density. Then we use the explicit asymptotic as in [Adl10, Thm 6.3.1].

Let us first consider the case that $(f(0), \nabla f(0), \nabla^2 f(0))$ having density in $\mathbb{R}^{(d+1)+d(d+1)/2}$. As mentioned, we'll use the following theorem by Adler

Theorem 3.3.5 (c.f. [Adl10] Theorem 6.3.1)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a stationary, C^2-smooth Gaussian field such that $(f(x), \nabla f(x), \nabla^2 f(x))$ is non-degenerate for all $x \in \mathbb{R}^d$. Further assume that $f(x)$ has zero mean, unit variance. Let $M_u(f,S)$ denote the number of local maxima of f in $S \subset \mathbb{R}^d$ with $f > u$. Then,*

$$\mathbb{E}[M_u(f,S)] = \frac{vol(S)\det(\Lambda_f)^{1/2}u^{d-1}}{(2\pi)^{(d+1)/2}} \exp\left(-u^2/2\right)(1+O(u^{-1}))$$

where Λ_f is the covariance matrix of ∇f and $O(u^{-1})$ is independent of choice of S.

Then by above theorem, for any Borel set $B \subset \mathbb{R}^d$

$$\begin{aligned} \mathbb{E}[\Phi_R(B)] &= \mathbb{E}[\eta_R(\mu(R) \cdot B)] \\ &= \frac{\mathrm{vol}(R\mu(R) \cdot B)}{(2\pi)^{(d+1)/2}} u^{d-1} \exp(-u^2/2)(1 + O(u^{-1})) \\ &= \mathrm{vol}(B)(1 + O(u^{-1})) \\ &\to \mathbb{E}[\Phi(B)] \quad \text{as } R \to \infty. \end{aligned} \tag{3.5}$$

Here, we've used the fact that determinant of covariance matrix of ∇f is 1, which follows from point 4 of Assumption 3.2.1.

Now we consider the monochromatic random waves (MRW) case. Let $f : \mathbb{R}^d \to \mathbb{R}$ be an MRW field. Let $\epsilon > 0$ and consider a random variable N, independent of the field f, which is standard normal variate. Define,

$$f_\epsilon(x) := f(x) + \epsilon N, \quad x \in \mathbb{R}^d.$$

Observe that f_ϵ is still a centred, stationary field and that $f_\epsilon(0), \nabla f_\epsilon(0), \nabla^2 f_\epsilon(0)$ is a Gaussian vector with density. Define $M(u, g)$ to be the number of local maxima of a Gaussian field g in $[0, 1]^d$.
Now we have, by an application of Kac-Rice formula,

$$\mathbb{E}[M(u, f_\epsilon)] = \int_{\mathbb{R}} \mathbb{E}[M(u - \epsilon b, f) | N = b] \phi(b) d\mathrm{b}.$$

where ϕ is the pdf of standard normal variate. Also,

$$M(u - \epsilon b, f) \longrightarrow M(u, f) \quad \text{a.s.} \quad \text{as} \quad \epsilon \to 0.$$

We know that $M(u, f)$ is integrable, and monotonic w.r.t. u, so using dominated convergence theorem,

$$\mathbb{E}[M(u - \epsilon b, f)] \to \mathbb{E}[M(u, f)], \quad \epsilon \to 0.$$

Since $\mathbb{E}[M(u, f)]$ is uniformly bounded in u, apply DCT for $\mathbb{E}[M(u-\epsilon b, f)]\phi(b)$ to get,

$$\lim_{\epsilon \to 0} \mathbb{E}[M(u, f_\epsilon)] = \mathbb{E}[M(u, f)].$$

Computing $\mathbb{E}[M(u, f_\epsilon)]$ is handled again by [Adl10, Thm 6.3.1] as eq (3.5).

Chapter 4

Current work

This was a follow up write up requested my Confirmation examiners (Jan Obloj and Zhongmin Qian).

Distance between high critical points of smooth Gaussian fields and Poisson point process

Following notations, conventions, assumptions from Chapter 3 of my confirmation report, we summarise the result mentioned in the confirmation viva under 'current work' section. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a C^2-smooth stationary Gaussian field with covariance kernel $K(x) = \mathbb{E}[f(0)f(x)]$ (assume correlation decay fast enough). Let Φ_R denote critical points of f in $[0, R]^d$ above increasing level $u(R)$, scaled appropriately down to $[0, 1]^d$ to have unit density.

Let ρ denote Wasserstein 1-distance on probability measures on the metric space $(\mathcal{M}, g)$, where $\mathcal{M}$ is the space of non-negative integer valued finite measures on $[0, 1]^d$ with metric g being Euclidean distance between point configurations. Let $\Phi_R^x, x \in [0, 1]^d$ denote the point process of conditioning of Φ_R to have a point at x, then removing the atom at x. This corresponds to the reduced Palm distribution of Φ_R, since it is a simple point process (see [Kal17, Chapter 6] for definition of Palm distribution and its properties). Now [CX04, Thm 3.4] implies that, roughly speaking, $\rho(\Phi_R,\ \mathrm{Pois}(Leb))$ is bounded above by the expected distance g between the configurations Φ_R and Φ_R^0. From here we can show that, for large enough R and a universal constant C

$$\rho(\Phi_R,\ \mathrm{Pois}(Leb)) \leq Cu(R) \max_{\|x\| \geq e^{u^2/2}} |\nabla K(x)|.$$

We are currently writing this result carefully and checking for gaps/errors. We will make this result (and the weak convergence result in Chapter 3 of confirmation) available on arXiv as soon as possible.

PhD timeline

We will try to submit the above Poisson convergence result to a journal by Dec/Jan. Next project I'm focusing on is the higher dimension case, to study asymptotic critical point structure of fields defined on $\mathbb{R}^d$ when $d \to \infty$. I will try to finish this project by February (I feel this is reasonable). Another project is to investigate the filament structure of random plane waves model (see https://users.flatironinstitute.org/ ahb/rpws/ for details). This is definitely more challenging project than the previous one. I regularly think about many toy problems from percolation theory of Gaussian fields which have potential to become a good PhD project, if progress is made.

From January, I'll start writing my PhD thesis, especially rewriting the intro in confirmation, expanding it significantly by making it a short survey style chapter. For main chapters, I'll add more examples and explanation to make it easier to read. My target is to finish writing the thesis by end of May 2025, giving myself 5 months to write.

Chapter 5

Future work

Many interesting questions arise out of our work and we list some of them below. We provide a brief motivation/background and a possible strategy towards the solution.

Filament structure of RPW

See Figure 5.1. What explains the apparent filament structure of the landscape of random plane wave (RPW)? Is it just a numerical artefact? Our result in the previous chapter establishes that there's no structure for local maxima for RPW at high levels. On the other hand, all critical points of RPW has a rigid structure, but not repulsive at small scale, and quite different from Poisson process [BCW19]. Looking at another criteria, Tacey [Tac23] showed that L^2 norm restricted to *any* long line is very close to that of the entire domain, which suggests that L^2 norm is not concentrated on any line. But this was kind of expected since these filaments spread in all directions. Currently, we're looking at the large scale limit of sum of delta measures at local maxima of RPW weighted with height.

Structure of critical points

A natural question which arises out of Chapter 4 is that of rate of convergence of law of high local maxima to Poisson process. It is easy to see that faster the rate of level going to infinity, the closer the distribution is to Poisson process. The rate should also depend on the covariance structure of the field, sharper the decay of correlation faster the convergence to Poisson process. The question is to quantify this in some metric like total variation distance or Kantorovich-Rubinstein (KR) distance. A plausible strategy would be to use Stein's method for approximating Poisson process, like that in [BSY22, Thm 3.1]. This method has been applied to problems like k-nearest neighbor ball of PPP. Here, the distance between a point process and PPP is bounded by

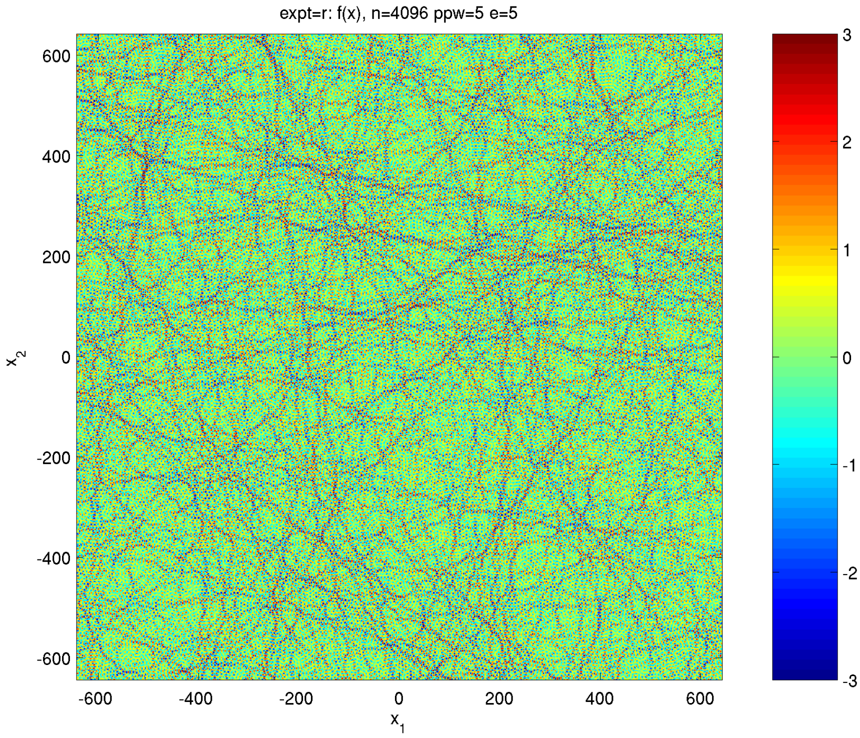


Figure 5.1: Filament structure of random plane wave. Picture by Alex Barnett

'distance' between a coupling that process and its reduced Palm version of it. See [BHJ92] for more on this.

Another interesting direction is to consider hole probabilities for critical points of Gaussian fields, i.e. what is the distribution of the critical points conditioned that there's no critical point in a large domain U. Interesting phenomenon like crowding of the points near the boundary of U has been observed for Ginibre ensemble [AR17]. Does similar thing happen for rigid field like RPW? Does Bargmann-Fock field mimick the behaviour of PPP?

Distance of nodal lengths

What is the optimal exponent of σ_D in RHS of Thm 2.2.2? Can we get a similar estimate for higher moments? Intuition from geometric analysis says we can derive the following bound,

$$|L_1 - L_2| \leq \| f_1 - f_2 \|_{C^2} \int_{f_1^{-1}(a)} \frac{\kappa}{|\nabla f_1|} ds + \text{ second order terms in } \| f_1 - f_2 \|_{C^2}$$

where κ is the mean curvature. Now, we may want to show that

$$\mathbb{E}\left[\left(\int_{f_1^{-1}(a)} \frac{\kappa}{|\nabla f_1|} ds\right)^m\right] < \infty$$

and hence get bounds on $\mathbb{E}|L_1 - L_2|^m$. We can use [AW09, Theorem 6.10] kind of result for higher moments. We also have that n- point correlations of curvature are finite even though $\mathbb{E}[\kappa^2] = \infty$.

Now, we try to prove the formula above. Say we have a perturbation of the field f, $f_t(x) := f(x) + tp(x)$ where $p(x)$ is thought to be small. When you have a coupled field f_1, then the difference is just $p(x)$. Consider the following flow of level lines of f_t in the normal direction (the speed of the flow depends on the inverse of norm of gradient of the function). For more info on this flow, refer [BMM22, Appendix A]

$$\frac{dx_t}{dt} = p(x_t)\frac{\nabla f_t(x_t)}{|\nabla f_t(x_t)|^2}$$

In this flow, the value $f_t(x_t)$ stays constant. Assume that, till $t = 1$, the flow hits no critical points of f (control this via quantitative Bulinskaya lemma). Now apply the first variation of area formula (See Wikipedia) Since mean curvature field is normal to the submanifold and the above flow is also normal flow, we have the following estimate,

$$\frac{d\mathrm{vol}(S_t)}{dt}\,|_{t=0} = \int_{S_0} p(x)\frac{\kappa(x)}{|\nabla f(x)|} dS(**)$$

where $S_t := f_t^{-1}(0)$. The second term on RHS of the given formula in Wikipedia vanished because the flow is in the normal direction, hence the projection onto the tangent space is zero. This "proves" point 3 above only in the limiting case where $\|f_1 - f_2\|$ is zero. Now, we need to integrate $(**)$ to get an actual difference of measure of level sets.

To apply the first variational formula, we need perturbing fields to be compactly supported, which means we have to mollify the edges of the vector field to zero. But then, you won't get the actual difference of length because the level sets are 'frozen' when the vector field is zero. So, in practise, we have to control the vector field even outside the box (which is non-compact) hence it is hard!

On the other hand, if you consider the geometric observable " total volume of level sets whose nodal components stay completely inside the box" then we don't have to do the 'non-compact' analysis as explained above. To study the boundary effect, we can borrow results from [BMW19].

Some of the applications of these estimates include a CLT for measure of level sets in increasing boxes.

Appendix A

Gaussian fields estimates

A.1 Method of comparison

One of the basic questions which pops up regularly when studying excursion of Gaussian fields is the following: given two Gaussian vectors in $\mathbb{R}^d$ with close enough covariance matrix, how close are the excursion probabilities. Here we recall a generalisation of the classic Berman's inequality, taken from [Pit96].

Say we're given n sequence of real numbers, that we call *discritising levels*,

$$\mathbf{u}(k) = (\cdots < u_{-1}(k) < u_0(k) < u_1(k) < \cdots) \quad k = 1, 2, 3, \ldots n.$$

Consider the $\sigma-$algebra $\mathcal{U}$ generated by the n-dim rectangles

$$\Pi_{\mathbf{i}} = \{(x_1, x_2, \ldots, x_n) : x_k \in [u_{i_k}, u_{i_k+1}(k)\}$$

where $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{Z}^n$ is a multi-index. Let $\mathbf{X}_0 = (X_0(1), X_0(2), \ldots, X_0(n))$ and $\mathbf{X}_1 = (X_1(1), X_1(2), \ldots, X_1(n))$ be two independent Gaussian vectors in $\mathbb{R}^n$ with zero mean. Consider an interpolation of these vectors,

$$\mathbf{X}_h = \sqrt{h}\mathbf{X}_1 + \sqrt{1-h}\mathbf{X}_0 \quad 0 \leq h \leq 1.$$

Denote by $R_h = \{r_h(i,j) : 1 \leq i, j \leq n\}$ the covariance matrix of $\mathbf{X}_h$.

Theorem A.1.1 (Thm 1.2, [Pit96]). *With notations as above, if $r_0(k,k) = r_1(k,k)$ for all k and $|r_0(k,l)| < 1$ for $k \neq l$, then for any $B \in \mathcal{U}$, we have*

$$|\mathbb{P}(\mathbf{X}_0 \in B) - \mathbb{P}(\mathbf{X}_1 \in B)| \leq 2 \sum_{k>l}^{n} |r_0(k,l) - r_1(k,l)| \sum_{i,j} \int_0^1 \phi(u_i(k), u_j(k); r_h(k,l)) dh$$

where $\phi(x, y; r)$ is the density of 2-dim Gaussian with covariance r.

A.2 Asymptotic excursion probability

In this section, we state a result on asymptotic excursion probability of stationary smooth Gaussian fields. Let $X : \mathbb{R}^d \to \mathbb{R}$ be a zero mean, unit variance, stationary C^2-smooth Gaussian field. Further assume that $(X(s), \nabla X(s))$ is non-degenerate Gaussian vector.

Theorem A.2.1 (Thm 7.1,[Pit96])**.** *Let $r(t,s)$ be the covariance function of the field X such that $r(t,s) < 1$ for $t \neq s$. Let $A \subset \mathbb{R}^d$ be a Jordan set of positive measure. Then,*

$$\mathbb{P}\left(\max_{t\in A} X(t) > u\right) = C\,vol(A)u^{d-1}\Psi(u)(1+o(1)) \qquad \text{as} \quad u \to \infty.$$

Here, the constant C depends only on the field and not on level u, $1-\Psi$ is cdf of standard Gaussian.

A.3 Maximum of Gaussian fields

It is a classical fact in probability that expected maximum of n i.i.d. standard Gaussian random variables behaves asymptotically like $\sqrt{2\log n}$ as $n \to \infty$. Also, it can shown easily that, even if the random variables are dependent, it cannot exceed $\sqrt{2\log n}$. What is bit surprising is that large number of Gaussian fields models with correlation decay 'fast enough' also have exact asymptotic $\sqrt{2\log n}$. Examples include 2-dim discrete Gaussian free field, energy landscape of Sherrington-Kirkpatrick model etc [Cha16]. For asymptotic distribution of the (centered, normalised) maximum, they are expected to converge to Gumbel distribution.

We have the same asymptotic for stationary smooth Gaussian fields.

Theorem A.3.1 (Thm 14.1, [Pit96])**.** *Let $X : \mathbb{R}^d \to \mathbb{R}$ be centered, unit variance, C^2-smooth Gaussian field with covariance $r(t) = \mathbb{E}[X(0)X(t)]$. Assume that, for some $\alpha > 0$,*

$$\int_{\mathbb{R}^d} |r(t)|^\alpha dt < \infty.$$

Then,

$$\mathbb{P}\left(\max_{t\in[0,R]^d}(X(t) - l_R)l_R < x\right) = \exp(-\exp(-x))$$

where l_R is the largest solution of the equation

$$\frac{R^d \det(\Lambda_X)^{1/2}}{(2\pi)^{d-1}} l^{d-1}\exp(-l^2/2) = 1$$

and Λ_X is the covariance matrix of $\nabla X(0)$.

From this theorem we can get exact asymptotic of the expected mean,

$$\frac{\mathbb{E}[\max_{t\in[0,R]^d} X(t)]}{\sqrt{2d\log R}} \to 1 \qquad \text{as} \quad R \to \infty.$$

Appendix B

Basic tools

B.1 Kac-Rice formula

Kac-Rice formulas are one of central tools in studying random fields, which helps us in computation of local observables. For simplicity, consider one dimensional smooth field f and we're counting number of zeros of the field in a given bounded interval $I = [a, b]$. Let $x_0 \in (a, b)$ be a zero and a regular point of f, i.e. $f'(x_0) \neq 0$. Given $\epsilon > 0$, for a sufficiently small neighborhood U of x_0, we have

$$\frac{1}{2\epsilon}\int_U |f'(x)|\mathbb{1}_{[|f(x)|\leq\epsilon]}dx = 1$$

which follows from fundamental theorem of calculus. Now, assuming all zeroes of f are regular and $f(a)f(b) \neq 0$, the number of zeros $N_I(f)$ is given by

$$N_I(f) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{[a,b]} |f'(x)|\mathbb{1}_{[|f(x)|\leq\epsilon]}dx$$

Now applying the same technique to a random field f, taking expectation, we have the following.

$$\begin{aligned}\mathbb{E}\left[N_I(f)\right] &= \mathbb{E}\left[\lim_{\epsilon\to 0}\int_{[a,b]} |f'(x)|\frac{1}{2\epsilon}\mathbb{1}_{[|f(x)|\leq\epsilon]}\right] dx \\ &= \lim_{\epsilon\to 0}\left[\int_{[a,b]}\mathbb{E}\left[|f'(x)|\frac{1}{2\epsilon}\mathbb{1}_{[|f(x)|\leq\epsilon]}\right] dx\right] \\ &= \int_{[a,b]}\mathbb{E}[|f'(x)||f(x)=0]p_{f(x)}(0)dx\end{aligned} \tag{B.1}$$

where p_X is the pdf of a Gaussian variable X. The integral, limit and expectation swaps can be justified when the random field follows some regularity. Even the higher moments can be computed by similar techniques.

Theorem B.1.1 (Gaussian Rice formula, ([AW09], Theorem 3.2))**.** *Let f be a C^1-smooth Gaussian field on an interval I. Let k be a positive integer. Assume* *that for every k pairwise distinct points $t_1, t_2, \ldots, t_k$ the joint distribution of $f(t_1), f(t_2), \ldots, f(t_k)$ does not degenerate. Then,*

$$\mathbb{E}[N_I^{[k]}] = \int_{I^k} \mathbb{E}\left[\prod_{i=1}^{k} |f'(t_i)| | f(t_1), f(t_2), \ldots f(t_k) = 0\right] p_{f(t_1), f(t_2) \ldots f(t_k)}(0) \prod_{i=1}^{k} dt_i$$

where $m^{[k]} = m(m-1)\cdots(m-k+1)$ given $m \geq k$, and 0 otherwise.

Observe that $f'(x)$ is also a Gaussian field and its regularity depends on that of the field f. Also we know that $(f'(t_1), f'(t_2), \ldots, f'(t_k), f(t_1), f(t_2) \ldots, f(t_k))$ is a Gaussian vector. Hence the conditional expectation in the above theorem can be explicitly computed using the covariance kernel and its derivatives. Note that the formula is particularly simpler when the field is stationary. We have,

$$\mathbb{E}[N_I] = \mathrm{Vol}(I)\mathbb{E}|f'(0)|/\sqrt{2\pi}$$

where the field is normalised to be $\mathrm{Var}f(0) = 1$.

There are many directions where we can generalise theorem B.1.1. We can consider fields in higher dimension, say $f : \mathbb{R}^d \to \mathbb{R}^k$ and ask for the expected geometric measure of the level sets of f when $k \leq d$. There are versions of Kac-Rice formula for non-Gaussian fields. We can generalise it to fields on a Riemannian manifold as well. In this article, we used the expected the Kac-Rice formula for expected lengths of level lines.

Kac-Rice formulas are also important in the analysis of critical points. Geometry (and topology) of level/excrusion sets crucially depend on the distribution (in a deterministic sense) of critical points of the field (see Morse theorems in classical topology). Also in many cases, even for non-local functionals like number of connected components of level sets, a pretty good estimates can obtained from looking at the critical points.

One interesting connection to random matrix theory for computing expected number of critical points was made by Fyodorov [Fyo04] in the context of theory of spin glsses. Note that by Kac-Rice formulas, expected number of critical points can be computed by an integral of conditional expectation of Hessian of the field. The novel idea of Fyodorov was to express law of Hessian in terms of a Gaussian Orthonormal Ensemble (GOE) matrices, where explicit computations are available.

To give an example, we quote a proposition from Cheng and Schwartzmann [CS18]. Let X be an isotropic Gaussian field on $\mathbb{R}^d$ and $\mu(X)$ be number of local maxima of X inside a unit volume ball. Then, under certain conditions on X, we have

$$\mathbb{E}[\mu(X)] = \Gamma((d+1)/2)\frac{2^{(d+1)/2}}{\pi^{(d+1)/2}\eta^N}\mathbb{E}_{GOE}^{N+1}\left[\exp(-\lambda_{d+1}^2/2)\right]$$

where expectation on RHS is w.r.t to GOE $(d+1) \times (d+1)$ ensembele, λ_{d+1} is the maximum eigenvalue of GOE, η is an explicit constant of the field X.

Refer chapters 3 and 6 of [AW09] or [AT09] for more on Kac-Rice formulas.

B.2 Bulinskaya lemma

While studying nodal geometry of smooth fields, it is desirable that nodal sets are stable under small perturbation. The particular bad event we want to get rid of is the event that the random Gaussian function and its gradient are simultaneously small at some point.

Lemma B.2.1. *Let U be an open set in $\mathbb{R}^n$ and let $g : U \to \mathbb{R}^{n+1}$ be a random function. Assume that $g \in C^1(U)$ a.s. and that the vector $g(x)$ has a density on $\mathbb{R}^{n+1}$ that is bounded uniformly over x in compact subsets of U. Then $g^{-1}(0)$ is almost surely empty.*

Now, applying the above lemma to C^1-Gaussian field $(f, \nabla f)$ and assuming it has a density (i.e. non-degeneracy of f), we get that, almost surely, 0 is a regular value of f. One immediate consequence of this is, we know that nodal sets are submanifolds, almost surely.

In [NS16], Nazarov and Sodin stated a quantitative version of Bulinskaya's lemma. This is helpful in bounding the probability of "bad events".

Lemma B.2.2. *Let $f : U \to \mathbb{R}$ be a C^2-smooth Gaussian field which is non-degenerate. Fix a compact subset $C \subset U$. Given $\delta > 0$, there exists $\tau > 0$ (possibly depending on C) such that*

$$\mathbb{P}\left(\min_{x \in C} \max\{|f(x)|, |\nabla f(x)|\} < \tau\right) < \delta.$$

B.3 Borell-TIS inequality

Theorem B.3.1 ([AT09],Theorem 2.1.1)**.** *Let $f(x)$ be a Gaussian process on D. Assume that the process is almost surely bounded on D. Define $||f|| = ||f||_D = \sup_D f(x)$. Then $\mathbb{E}[||f||] < \infty$, and*

$$\forall u > 0,\ \mathbb{P}(||f|| - \mathbb{E}[||f||] > u) \leq e^{-u^2/2\sigma_D^2}$$

where $\sigma_D^2 = \sup_D \mathbb{E}[f(x)^2]$

This theorem holds for continuous Gaussian process, but in the smooth setting we are guaranteed that on any bounded D, the supremum is a.s. finite, due to Kolmogorov's theorem.

Using Borell-TIS inequality, we can bound the tail probability of C^k-norm of a smooth Gaussian field, see [BM22, Lemma 2.1] for example. Fixing the law of the field f and the domain D, $\mathbb{P}(||f||_{C^k} > l)$ is essentially bounded by $\exp\left(-c \cdot l^2\right)$ where c depends on variances of f and its derivatives on D. This is bit surprising, since the tail of single Gaussian random variable also has a similar upper bound.

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