Level sets of smooth Gaussian fields

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Motivation example

In 1977 M. Berry conjectured that high energy eigenfunctions in the chaotic case have statistically the same behaviour as random plane waves. (Figures from Bogomolny-Schmit paper)

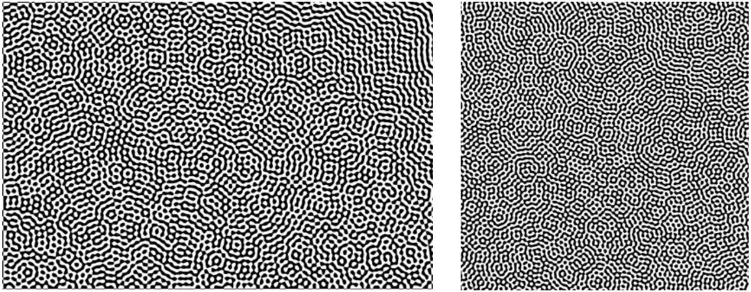


Figure: Left: nodal portrait of the eigenfunction of a quarter of the stadium with energy E = 10092.029. Right: Snapshot of a random wavefunction with wavenumber 100 ($\simeq \sqrt{E}$)

Smooth Gaussian fields

- Let $V \in \mathbb{R}^d$ be an open set. A C^k-*smooth Gaussian field* is a Gaussian process indexed by V which has C^k-smooth sample paths. [Fields are centered throughout this talk]
- **Kolmogorov theorem**: Suppose that $K : V \times V \to \mathbb{R}$ is a positive definite symmetric function of class $C^{k,k}(V \times V)$ and, in addition, that

$$N := \max_{|\alpha|,|\beta| \leq k} \sup_{x,y \in V} |\partial_x^\alpha \partial_y^\beta K(x,y)| < \infty.$$

Then there exists a (unique up to an equivalence of distribution) C^{k-1} Gaussian function f on V with the covariance kernel K. Moreover, $\mathbb{E}||f||_{C^{k-1}} \leq C\sqrt{N}$.

Stationary fields

- Call a Gaussian field on $\mathbb{R}^d$ *stationary* or *translation invariant* if its covariance kernel $K(x, y)$ depends only on $x - y$, say $K(x, y) = k(x - y)$.
- **Bochner theorem**: For a continuous k, k is a Fourier transform of a finite symmetric ($\rho(A) = \rho(-A)$) positive Borel measure ρ on $\mathbb{R}^d$, i.e.

$$k(x) = \int_{\mathbb{R}^d} e^{2\pi i(\lambda \cdot x)} d\rho(\lambda).$$

- Call ρ the *spectral measure* of the field.
- The field is a fourier transform of white noise on ρ, i.e.

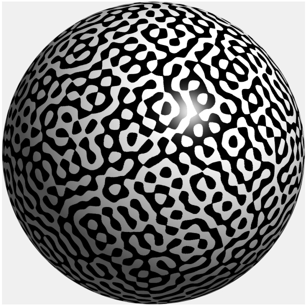
$$f(x) = W_\rho(e^{2\pi i x \cdot t})$$

The properties of f, K, ρ are closely related.

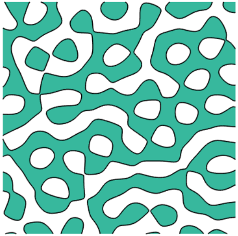
Examples: Random plane waves

- Spectral measure is (normalised) arc length measure on $S^1 \subset \mathbb{R}^2$. So covariance kernel is $J_0(|x-y|)$, where J_0 is zeroth Bessel function. Here the covariance function oscillates around zero, and decays like $|x-y|^{-1/2}$
- Local scaling limit of a number of other Gaussian fields. E.g. Random spherical harmonics [Wig22].

Pictures



(a) Random spherical harmonic of high degree



(b) A closer look at random plane wave nodal lines

Figure: Both pictures by Dmitry Beliaev

Examples: Bargmann-Fock field

- Covariance kernel is $K(x,y) = e^{-|x-y|^2/2}$. Hence the spectral measure has Gaussian-type density.
- The field can be written as

$$f(x) = e^{-|x|^2/2} \sum_{n,m \geq 0} \frac{a_{n,m}}{\sqrt{n!m!}} x_1^n x_2^m$$

- Thought of as a limit of Gaussian ensemble of homogeneous polynomials. So zero sets are "portrait of 'typical' algebraic variety".
- Many percolation theoretic properties are easier to establish in this model because the correlation decay is very fast.

Pictures

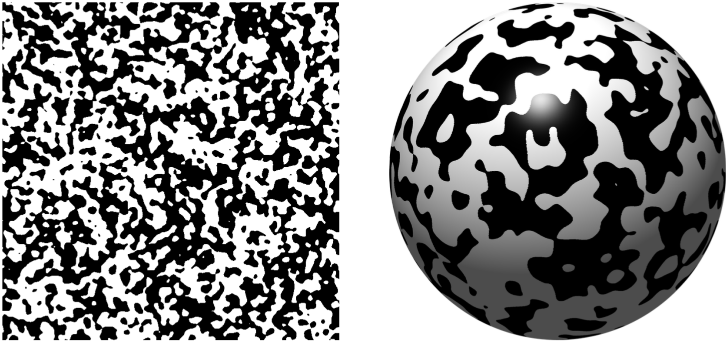


Figure: (Left) Bargman-Fock field sample. (Right) Gaussian ensemble of homogeneous polynomials of degree 300. The scale is $d^{-1/2}$ where d is the degree. Picture: Dmitry Beliaev

Questions

We're interested in large scale geometry of level/excursion sets $\{f = l\}$ or $\{f \geq l\}$. Quantities that we're interested in:

1. Local
   - Volume of level sets
   - Critical point structure of the field
2. Non-local
   - Number of components of level sets
   - Percolation theoretic probabilities (like box crossing)

Volume of level sets

Given two coupled smooth Gaussian fields f_1, f_2 which are 'close' to each other, how close are their volume of level sets on average?

We give an upper bound in terms of average C^2-fluctuation of the field $F = f_1 - f_2$.

Setup

Let $D \subset \mathbb{R}^d$ be a regular enough domain. Consider two coupled Gaussian fields f_1, f_2 which are

1. C^2-smooth,
2. stationary (i.e. two-point correlation function $r(x, y)$ is translation invariant),
3. non-degenerate ($(f(0), \nabla f(0))$ has density in $\mathbb{R}^{d+1}$).

Let $F = f_1 - f_2$ and define its average C^2-fluctuation in D as

$$\sigma_D^2 = \sup_{x \in D} \sup_{|\alpha| \leq 2} \operatorname{Var}\left(\partial^\alpha F(x)\right).$$

Note that F might not be stationary, even if f_1, f_2 are stationary.

Result

Let L_1, L_2 denote $(d-1)$-dimensional Hausdorff measures of $\{f_1^{-1}(0) \cap D\}, \{f_2^{-1}(0) \cap D\}$ respectively.

Theorem (Beliaev, H. 2023)

With the setup as above, we have

$$\mathbb{E}|L_1 - L_2| \leq C(D)\sigma_D^{1/7}$$

given that $\sigma_D < 1$. Here $C(D) = c \cdot \mathrm{vol}(D)\sqrt{\log(\mathrm{vol}(D))}$ where c is a constant depending only on laws of the fields, but not the coupling.

Comments

1. The constant c in the above theorem is fairly explicit, it depends on things like $\mathbb{E}[|\partial_x^2 f_1(0)|^2]$.
2. Bounding σ_D is also amenable, E.g. if ρ_1, ρ_2 are spectral measures of f_1, f_2 then

$$\sigma_D^2 \leq \tilde{c}\mathrm{vol}(D) \inf_{\rho \in \Pi(\rho_1, \rho_2)} \int (|s|^2 + |t|^2 + 1)^{d+1} |s - t|^2 \, \mathrm{d}\rho(s, t).$$

   See [BM22, Thm 4.1] for an explanation.

Proof heuristics

From geometric analysis, we know

rate of change of volume of level set of f = mean curvature of the submanifold $f^{-1}(0)$.

For a function f, applying divergence theorem for the vector field $\nabla f / |\nabla f|$ on $f^{-1}[a, b]$ we have the following proposition.

Proposition

Let $L(t) = vol^{d-1}(f^{-1}(t))$, then for regular enough f

$$L(b) - L(a) = \underbrace{\iint_D \kappa \mathbb{1}_{f \in [a,b]} dVol}_{\text{Bulk term}} + \text{Boundary term}.$$

Differentiation of nodal volumes

Fix a bounded domain $D \subset \mathbb{R}^d$, and a Gaussian field f. Let $V(f)$ denote the volume of $f^{-1}(0) \cap D$. Denote by $\mathbb{D}^{k,p}$ the *Malliavin-Sobolev space* (k-times differentiable with derivatives having p moments).

- (Peccati-Stecconi '24) For dimensions $d = 2, 3$, we have $V(f) \in \mathbb{D}^{1,1}$ and for $d \geq 4$, $V(f) \in \mathbb{D}^{1,2}$.
- Existence of absolute continuous part of $V(f)$ w.r.t Lebesgue measure.

Proof hueristic is the same as before.

Further questions

1. Optimal exponent of σ in the theorem? Get a matching lower bound.
2. Prove similar estimates for higher moments. (Challenge: $\mathbb{E}[\kappa^2] = \infty$)

Intuition from "First variation of area formula" from geometric analysis is the following:

$$\mathbb{E}[|L_1 - L_2|] \simeq \sigma_D \cdot \mathbb{E} \int_0^1 \int_{F_t^{-1}(0)} \frac{|\kappa_t(x)|}{|\nabla F_t(x)|} dS(x) dt$$

where $F_t(x) = f_1(x) + t(f_2(x) - f_1(x))$.

Next project: critical points structure

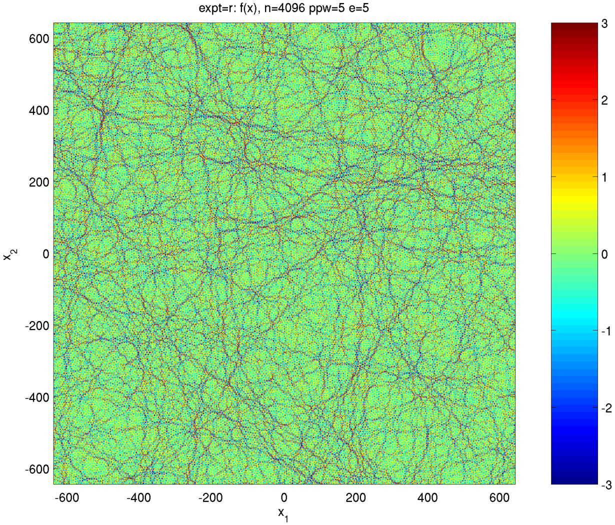


Figure: Filament structure of random plane wave. Picture by Alex Barnett

Thank you!

References

[BM22] D. Beliaev and R. W. Maffucci. "Coupling of stationary fields with application to arithmetic waves". In: *Stochastic Processes and their Applications* 151 (2022), pp. 436–450. ISSN: 0304-4149. DOI: https://doi.org/10.1016/j.spa.2022.06.009.

[Wig22] I. Wigman. "On the nodal structures of random fields – a decade of results". In: *arXiv.2206.10020* (2022). Publisher: arXiv. DOI: 10.48550/ARXIV.2206.10020.