Analysis of the Aw-Rascle model of traffic

Ewelina Zatorska

Imperial College London

"With a Little Help from My Friends"

N. Chaudhuri, P. Gwiazda,

N. Chaudhuri, E. Feireisl,

N. Chaudhuri, L. Navoret, C. Perrin

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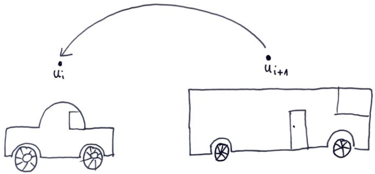
About the Aw-Rascle model

Models of the one-lane traffic

In one-lane traffic it is clear who the leader is



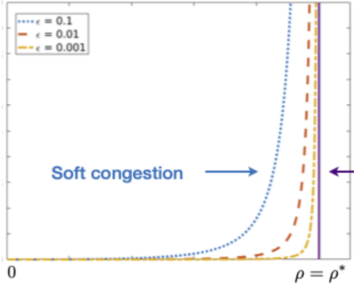
From Follow the Leader to fluid-like Aw-Rascle system:

$$\begin{cases} \dot{x}_i = u_i, \\ \dot{u}_i = C \frac{u_{i+1} - u_i}{(x_{i+1} - x_i)^{\gamma+1}}, \end{cases} \longrightarrow \begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0, \\ \partial_t(\varrho w) + \partial_x(\varrho w u) = 0, \\ w = u + P(\varrho) \end{cases}$$

where $P(\varrho) = \varrho^\gamma$ is the cost (offset) function.

Aw, Klar, Rascle, Materne. Derivation of continuum traffic flow models from microscopic follow-the-leader models. *SIAM J. Math. Anal.*, 63(1):259–278, 2002.

Problem: Maximal velocity and maximal density constraints not preserved.



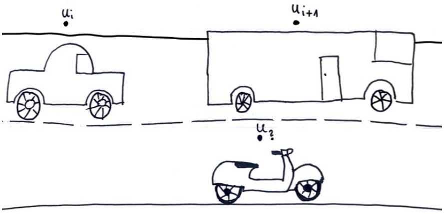
Solution: Cost function with maximal density constraint $\rho^* > 0$

$$P_\epsilon(\rho) = \epsilon \left(\frac{\rho}{\rho^* - \rho} \right)^\gamma .$$

F. Berthelin, P. Degond, M. Delitata, and M. Rascle. A Model for the Formation and Evolution of Traffic Jams. *ARMA*, 187, 185–220, 2008.

Multi-lane models

Who is the leader now?



One dimension $\longrightarrow$ Several dimensions

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0, \\ \partial_t(\varrho w) + \partial_x(\varrho w u) = 0, \end{cases} \longrightarrow \begin{cases} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ \partial_t(\varrho \mathbf{w}) + \mathrm{div}(\varrho \mathbf{w} \otimes \mathbf{u}) = 0. \end{cases}$$

Problem: Dimension incompatibility: $\underbrace{\mathbf{w}}_{\text{vector}} = \underbrace{\mathbf{u}}_{\text{vector}} + \underbrace{P(\varrho)}_{\text{scalar}}.$

Solutions to dimension incompatibility problem

Either:

$$\mathbf{w} = \mathbf{u} + \mathbf{P}(\varrho),$$

where $\mathbf{P}(\varrho) = [P_1(\varrho), P_2(\varrho)]$.

M. Herty, S. Moutari, G. Visconti. Macroscopic modeling of multilane motorways using a two-dimensional second-order model of traffic flow. *SIAM J. Appl. Math.* 78(4):2252–2278, 2018.

Or:

$$\mathbf{w} = \mathbf{u} + \nabla p(\varrho),$$

where $p(\varrho)$ is a scalar function.

A.Tosin, P. Degond, E. Zatorska Students' theses 2016-2017.

Observations about the model

- Taking the offset function $P(\varrho) = \partial_x p(\varrho) = \frac{\lambda(\varrho)}{\varrho^2}\partial_x \varrho$, we get pressureless, compressible, degenerate Navier-Stokes equations:

$$\partial_t \varrho + \partial_x(\varrho u) = 0,$$
$$\partial_t(\varrho u) + \partial_x(\varrho u^2) - \partial_x\left(\lambda(\varrho)\partial_x u\right) = 0.$$

- In more dimensions this dissipative effect looks differently

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) = \nabla_x(\varrho Q'(\varrho) \operatorname{div} \mathbf{u}) + \mathcal{L}[\nabla_x Q(\varrho), \nabla_x \mathbf{u}],$$

where $Q'(\varrho) = \varrho p'(\varrho)$ and

$$\mathcal{L}[\nabla_x Q(\varrho), \nabla_x \mathbf{u}] = \nabla_x(\nabla_x Q(\varrho) \cdot \mathbf{u}) - \operatorname{div}(\nabla_x Q(\varrho) \otimes \mathbf{u}),$$

which is a lower order term

$$(\mathcal{L}[\nabla_x Q(\varrho), \nabla_x \mathbf{u}])_j = \sum_{i=1}^{3} \left(\partial_{x_i} Q(\varrho)\partial_{x_j} u_i - \partial_{x_j} Q(\varrho)\partial_{x_i} u_i\right), \quad j = 1, 2, 3.$$

Existence and weak-strong uniqueness of measure-valued solutions

The set up of the problem

Let $\mathbf{w} = \mathbf{u} + \nabla p(\varrho)$ we can either solve:

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ \partial_t(\varrho \mathbf{w}) + \operatorname{div}(\varrho \mathbf{w} \otimes \mathbf{u}) = 0, \end{cases}$$

or equivalently:

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{w}) - \operatorname{div}(\sqrt{\varrho} \nabla Q) = 0, \\ \partial_t(\varrho \mathbf{w}) + \operatorname{div}(\varrho \mathbf{w} \otimes \mathbf{w}) = \operatorname{div}(\sqrt{\varrho} \nabla Q \otimes \sqrt{\varrho} \mathbf{w}), \end{cases}$$

where $Q'(\varrho) = \sqrt{\varrho} p'(\varrho)$.

We consider $\Omega = \mathbb{T}^d$ with the initial data $\varrho(0,x) = \varrho_0 \geq 0$, $(\varrho \mathbf{w})(0,x) = \mathbf{m}_0$, satisfying the energy bound

$$E_0 = \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + E(\varrho_0) \right) \mathrm{d}x < \infty, \quad \text{where} \quad E(\varrho) = \int_0^\varrho p(s)\, ds.$$

The uniform estimates are:

$$\|\sqrt{\varrho_n}\mathbf{w}_n\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} \leq C,$$
$$\|E(\varrho_n)\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} \leq C,$$
$$\|Q(\varrho_n)\|_{L^2(0,T;W^{1,2}(\mathbb{T}^d))} \leq C,$$

where $E(\varrho) = \int_0^\varrho p(s)\, ds$, $Q'(\varrho) = \sqrt{\varrho}p'(\varrho)$.

Remarks:

1. There is no uniform bound on $\mathbf{w}_n$.
2. The estimates for ϱ_n are quite strong.

- The continuity equation

$$\partial_t \varrho_n + \operatorname{div}(\underbrace{\sqrt{\varrho_n}\sqrt{\varrho_n}\mathbf{w}_n}_{L^\infty(L^p)}) - \operatorname{div}(\underbrace{\sqrt{\varrho_n}\nabla Q(\varrho_n)}_{L^2(L^p)}) = 0,$$

- The momentum equation

$$\partial_t(\sqrt{\varrho_n}\sqrt{\varrho_n}\mathbf{w}_n) + \operatorname{div}(\underbrace{\sqrt{\varrho_n}\mathbf{w}_n \otimes \sqrt{\varrho_n}\mathbf{w}_n}_{L^\infty(L^1)}) = \operatorname{div}(\underbrace{\nabla Q(\varrho_n) \otimes \sqrt{\varrho_n}\mathbf{w}_n}_{L^2(L^1)}).$$

Young measures

A Young measure is a measurable (weak*) map

$$\mathcal{V} : Q \subset \mathbb{R}^k \to \mathcal{P}(\mathbb{R}^N),$$

in the sense that

$$z \in Q \to \langle \mathcal{V}_z ; g(\xi) \rangle = \int_{\mathbb{R}^N} g(\xi) d\mathcal{V}_z(\xi)$$

is Borel measurable $\forall g \in C_0(\mathbb{R}^N)$.

Any measurable function $\mathbf{u}_n : Q \to \mathbb{R}^N$ generates a measure

$$\mathbf{u}_n : z \in Q \to \delta_{u_n(z)} \in \mathcal{P}(\mathbb{R}^N),$$

moreover $\mathbf{u}_n \to \mathcal{V}$ in the natural topology $L^\infty_{waek*}(Q; \mathcal{M}(\mathbb{R}^N))$, meaning that

$$\langle \mathbf{u}_n ; g(\xi) \rangle \to \langle \mathcal{V} ; g(\xi) \rangle \quad \text{weakly}^* \quad \text{in} \quad L^\infty(Q), \quad \forall g \in C_0(\mathbb{R}^N).$$

Definition: $\mathcal{V}$ *is called the Young measure generated by* $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$.

Oscillations and concentrations

Having a sequence $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ such that

$$\|\mathbf{u}_n\|_{L^1(Q)} \leq C \quad \text{and} \quad \|b(\mathbf{u}_n)\|_{L^p(Q)} \leq C, \quad p > 1$$

then $\lim_{n\to\infty} b(\mathbf{u}_n)$ can be characterised by $\mathcal{V}$, i.e.

$$\int_Q \phi(z) b(\mathbf{u}_n(z))\, dz \to \int_Q \phi(z)\langle \mathcal{V}_z; b(\xi)\rangle\, dz, \quad \forall \phi \in L^{p'}(Q).$$

But if $\|b(\mathbf{u}_n)\|_{L^1(Q)} \leq C$ only, then

$$b(\mathbf{u}_n) \to \overline{b(u)} \in \mathcal{M}(Q).$$

Remark: Only the oscillations are captured by the Young measure, the concentrations are not!

Definition: *We call*

$$\mathcal{R}_b = \overline{b(u)} - \langle \mathcal{V}_z; b(\xi)\rangle$$

a defect measure for function b.

Our definition of solution

Our Young measure is generated by the sequence $\{\varrho_n, \sqrt{\varrho_n}\mathbf{w}_n, \nabla Q(\varrho_n)\}_{n\in\mathbb{N}}$, and so we consider $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\mathbb{T}^d}$, and

$$\mathcal{V} \in L^\infty_{\text{weak-(*)}}\big((0,T)\times\mathbb{T}^d; \mathcal{P}(\mathcal{F})\big),$$

on the phase space

$$\mathcal{F} = \left\{ \left(\tilde{\varrho}, \widetilde{\sqrt{\varrho}\mathbf{w}}, \widetilde{D_Q}\right) \mid \tilde{\varrho} \in [0,\infty),\ \widetilde{\sqrt{\varrho}\mathbf{w}} \in \mathbb{R}^d,\ \widetilde{D_Q} \in \mathbb{R}^d \right\}.$$

Our convergence results allow us to identify

$$\varrho = \langle \mathcal{V}_{t,x}; \tilde{\varrho} \rangle, \quad \sqrt{\varrho}\left\langle \mathcal{V}_{t,x}; \widetilde{\sqrt{\varrho}\mathbf{w}} \right\rangle = \left\langle \mathcal{V}_{t,x}; \sqrt{\tilde{\varrho}}\widetilde{\sqrt{\varrho}\mathbf{w}} \right\rangle,$$

$$Q(\varrho) = \langle \mathcal{V}_{t,x}; Q(\tilde{\varrho}) \rangle, \quad \nabla_x Q(\varrho) = \left\langle \mathcal{V}_{t,x}; \widetilde{D_Q} \right\rangle.$$

In particular, we have

$$\mathcal{V}_{t,x} = \delta_{\{\varrho(t,x)\}} \otimes Y_{t,x} \quad \text{for a.a. } (t,x) \in (0,T)\times\mathbb{T}^d,$$

where $Y \in L^\infty_{\text{weak-(*)}}\big((0,T)\times\mathbb{T}^d; \mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d)\big)$.

Weak formulation

1. The continuity equation

$$\partial_t \varrho + \operatorname{div}\left(\sqrt{\varrho}\left\langle \mathcal{V}_{t,x}; \widetilde{\sqrt{\varrho}\mathbf{w}}\right\rangle\right) - \operatorname{div}(\sqrt{\varrho}\nabla_x Q) = 0$$

2. The momentum equation

$$\partial_t \left(\sqrt{\varrho}\left\langle \mathcal{V}_{t,x}; \widetilde{\sqrt{\varrho}\mathbf{w}}\right\rangle\right) + \operatorname{div}\left(\left\langle \mathcal{V}_{t,x}; \widetilde{\sqrt{\varrho}\mathbf{w}} \otimes \widetilde{\sqrt{\varrho}\mathbf{w}}\right\rangle\right)$$
$$- \operatorname{div}\left(\left\langle \mathcal{V}_{t,x}; \widetilde{\sqrt{\varrho}\mathbf{w}} \otimes \widetilde{D_Q}\right\rangle\right) + \operatorname{div}(r^M) = 0.$$

are satisfied in the sense of distributions, where

$$r^M \in L^\infty_{\text{weak-}(*)}(0,T;\mathcal{M}(\mathbb{T}^d;\mathbb{R}^{d\times d})) + \mathcal{M}([0,T]\times\mathbb{T}^d;\mathbb{R}^{d\times d}).$$

3. The energy inequality

$$\int_{\mathbb{T}^d} \left\langle \mathcal{V}_{\tau,x}; \frac{1}{2}\left|\widetilde{\sqrt{\varrho}\mathbf{w}}\right|^2 + E(\tilde{\varrho})\right\rangle \mathrm{d}x + \int_0^\tau\int_\Omega \left\langle \mathcal{V}_{t,x}; |\widetilde{D_Q}|^2\right\rangle \mathrm{d}x\mathrm{d}t + \mathcal{D}(\tau)$$
$$\leq \int_{\mathbb{T}^d} \left\langle \mathcal{V}_{0,x}; \frac{1}{2}\left|\widetilde{\sqrt{\varrho}\mathbf{w}}\right|^2 + E(\tilde{\varrho})\right\rangle \mathrm{d}x + \int_0^\tau\int_\Omega \left\langle \mathcal{V}_{t,x}; \widetilde{\sqrt{\varrho}\mathbf{w}} \cdot \widetilde{D_Q}\right\rangle \mathrm{d}x\mathrm{d}t + \int_{(0,\tau)\times\mathbb{T}^d} \mathrm{d}\mathcal{R},$$

where

$$\mathcal{R} \in L^\infty_{\text{weak-}(*)}(0,T;\mathcal{M}(\mathbb{T}^d)) + \mathcal{M}([0,T]\times\mathbb{T}^d).$$

4. The weights are compatible, i.e. $\mathcal{D} \equiv 0 \implies \mathcal{R}, r^M \equiv 0$.

Weak-strong uniqueness

Theorem (Gwiazda, Chaudhuri, Zatorska '22)

Let $(\mathcal{V}, \mathcal{D})$ *be a measure valued solution in* $(0, T) \times \mathbb{T}^d$ *of the Aw-Rascle system. Let* $(\bar{\varrho}, \bar{\mathbf{w}})$ *be a strong solution to the same system in* $(0, T) \times \mathbb{T}^d$ *with initial data* $(\bar{\varrho}_0, \bar{\mathbf{w}}_0) \in (C^2(\mathbb{T}^d), C^2(\mathbb{T}^d; \mathbb{R}^d))$ *satisfying* $\bar{\varrho}_0 > 0$. *We assume that the strong solution belongs to the class*

$$\bar{\varrho} \in C^1(0, T; C^2(\mathbb{T}^d)),\ \bar{\mathbf{w}} \in C^1(0, T; C^2(\mathbb{T}^d); \mathbb{R}^d) \text{ with } \bar{\varrho} > 0.$$

If the initial states coincide, i.e.

$$\mathcal{V}_{0,x} = \delta_{\{\bar{\varrho}_0(x), \bar{\mathbf{w}}_0(x)\}}, \text{ for a.e. } x \in \mathbb{T}^d$$

then $\mathcal{D} = 0$, *and*

$$\mathcal{V}_{\tau,x} = \delta_{\{\bar{\varrho}(\tau,x), \sqrt{\bar{\varrho}}\bar{\mathbf{w}}(\tau,x), \nabla_x Q(\bar{\varrho})(\tau,x)\}}, \text{ for a.e. } (\tau, x) \in (0, T) \times \mathbb{T}^d.$$

Nonuniqueness of weak solutions to the dissipative Aw-Rascle model

Multi-dimensional model again

Recall the main system and the offset function:

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\ \partial_t(\varrho \mathbf{w}) + \operatorname{div}(\varrho \mathbf{w} \otimes \mathbf{u}) &= 0, \\ \mathbf{w} = \mathbf{u} + \mathbf{P}(\varrho) &+ \nabla p(\varrho), \end{aligned}$$

where $\mathbf{P}(\varrho) = [P_1(\varrho), P_2(\varrho)]$ and $P_1(\varrho),\ P_2(\varrho),\ p(\varrho)$ might be singular functions of the density:

$$\approx \left(\frac{\varrho}{\varrho^* - \varrho}\right)^{\gamma}.$$

For simplicity we consider the problem on the torus:

$$\mathbb{T}^d = \left([-1,1]|_{\{-1;1\}}\right)^d, \ d = 2,3.$$

Ill posedness with respect to the initial–final data

Any initial density–velocity data $(\varrho_0, \mathbf{u}_0) = (\varrho(0, \cdot), \mathbf{u}(0, \cdot))$ can connect to arbitrary terminal state $(\varrho_T, \mathbf{u}_T) = ((\varrho(T, \cdot), \mathbf{u}(T, \cdot))$ via a weak solution.

More specifically, we consider

$$\varrho_0, \varrho_T \in C^2(\mathbb{T}^d), \quad \inf_{\mathbb{T}^d} \varrho_0 > 0, \quad \inf_{\mathbb{T}^d} \varrho_T > 0,$$

$$\int_{\mathbb{T}^d} \varrho_0 \, \mathrm{d}x = \int_{\mathbb{T}^d} \varrho_T \, \mathrm{d}x,$$

together with

$$\mathbf{u}_0, \mathbf{u}_T \in C^3(\mathbb{T}^d; R^d),$$

$$\int_{\mathbb{T}^d} \varrho_T \mathbf{u}_T \, \mathrm{d}x - \int_{\mathbb{T}^d} \varrho_0 \mathbf{u}_0 \, \mathrm{d}x = \int_{\mathbb{T}^d} \varrho_0 \mathbf{P}(\varrho_0) \, \mathrm{d}x - \int_{\mathbb{T}^d} \varrho_T \mathbf{P}(\varrho_T) \, \mathrm{d}x.$$

Theorem (Chaudhuri, Feireisl, Zatorska '22)

Let $d = 2, 3$. *Suppose that*

$$\mathbf{P} \in C^2((0,\infty); R^d), \ \ p \in C^2((0,\infty)).$$

Let $(\varrho_0, \mathbf{u}_0)$, $(\varrho_T, \mathbf{u}_T)$ *satisfy assumptions above.*
Then, the Aw-Rascle system, endowed with the periodic boundary conditions admits infinitely many weak solutions in the class

$$\varrho \in C^2([0,T] \times \mathbb{T}^d), \quad \mathbf{u} \in L^\infty((0,T) \times \mathbb{T}^d; R^d)$$

such that

$$\varrho(0,\cdot) = \varrho_0, \ \ \varrho(T,\cdot) = \varrho_T, \ \ (\varrho\mathbf{u})(0,\cdot) = \varrho_0\mathbf{u}_0, \ \ (\varrho\mathbf{u})(T,\cdot) = \varrho_T\mathbf{u}_T.$$

Satisfaction of the energy inequality

The AR system admits a natural energy functional

$$E(\varrho, \mathbf{u}) = \frac{1}{2}\varrho \left|\mathbf{u} + \mathbf{P}(\varrho) + \nabla p(\varrho)\right|^2 .$$

Given the periodic boundary conditions, the total energy of smooth solutions is conserved,

$$\int_{\mathbb{T}^d} E(\varrho, \mathbf{u})(t, \cdot)\ \mathrm{d}x = \int_{\mathbb{T}^d} E(\varrho_0, \mathbf{u}_0)\ \mathrm{d}x \text{ for any } t \in [0, T].$$

Admissible weak solutions should satisfy at least the energy inequality

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\mathbb{T}^d} E(\varrho, \mathbf{u})\ \mathrm{d}x \leq 0, \quad \int_{\mathbb{T}^d} E(\varrho, \mathbf{u})(t, \cdot)\ \mathrm{d}x \leq \int_{\mathbb{T}^d} E(\varrho_0, \mathbf{u}_0)\ \mathrm{d}x.$$

Theorem (Chaudhuri, Feireisl, Zatorska '22)

Let $d = 2, 3$. *Suppose that*

$$\mathbf{P} \in C^2((0,\infty); R^d),\ p \in C^2((0,\infty)).$$

Let $\varrho_0 \in C^2(\mathbb{T}^d)$, $\inf_{\mathbb{T}^d} \varrho_0 > 0$ *be given. Then there exists an initial velocity* $\mathbf{u}_0 \in L^\infty(\mathbb{T}^d; R^d)$ *such that the AR system, endowed with the periodic boundary conditions admits infinitely many weak solutions in the class*

$$\varrho \in C^2([0,T] \times \mathbb{T}^d), \mathbf{u} \in L^\infty((0,T) \times \mathbb{T}^d; R^d)$$

satisfying

$$\varrho(0,\cdot) = \varrho(T,\cdot) = \varrho_0,\ (\varrho\mathbf{u})(T,\cdot) = 0,$$

together with the energy inequality

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\mathbb{T}^d} E(\varrho,\mathbf{u})\ \mathrm{d}x \le 0,\ \int_{\mathbb{T}^d} E(\varrho,\mathbf{u})(t,\cdot)\ \mathrm{d}x \le \int_{\mathbb{T}^d} E(\varrho_0,\mathbf{u}_0)\ \mathrm{d}x.$$

Convex integration

- C. De Lellis and L. Székelyhidi, Jr.. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, 195(1):225–260, 2010.
- E. Chiodaroli. A counterexample to well-posedness of entropy solutions to the compressible Euler system. *J. Hyperbolic Differ. Equ.*, 11(3):493–519, 2014.
- E. Feireisl. Weak solutions to problems involving inviscid fluids. In *Mathematical Fluid Dynamics, Present and Future*, Volume 183 of *Springer Proceedings in Mathematics and Statistics*, pages 377–399. Springer, New York, 2016.
- T. Buckmaster and V. Vicol. Nonuniqueness of weak solutions to the Navier-Stokes equation. *Ann. of Math.* (2) 189(1): 101–144, 2019.
- R. M. Chen, A. F. Vasseur, C. Yu. Global ill-posedness for a dense set of initial data to the Isentropic system of gas dynamics. To appear in *Advances in Mathematics*, arXiv:2103.04905, 2021.

Asymptotic analysis of the dissipative Aw-Rascle model in 1D

Compressible and incompressible crowd dynamics

- Compressible Navier-Stokes/Euler equations (constant temperature):

$$\begin{cases} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \pi_\varepsilon(\varrho) = \mathbf{0}, \end{cases}$$

where the unknowns are ϱ and $\mathbf{u}$, and

$$\pi_\varepsilon(\varrho) = \varepsilon \left(\frac{\varrho}{\varrho^* - \varrho} \right)^\gamma, \quad \gamma > 0.$$

- Incompressibe Navier-Stokes/Euler equations:

$$\begin{cases} \mathrm{div}\, \mathbf{u} = 0, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = \mathbf{0}, \end{cases}$$

where the unknowns are $\mathbf{u}$ and π.

Mixed compressible-incompressible model of crowd

The two-phase system

$$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0$$
$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \pi = \mathbf{0}$$

$$\begin{cases} 0 \le \varrho \le \varrho^*, \\ \pi \ge 0, \\ (\varrho^* - \varrho)\pi = 0. \end{cases}$$

A. $0 \le \varrho < \varrho^*$ $\rightarrow$ pressureless Euler equations,

B. $\varrho = \varrho^*$ $\rightarrow$ incompressible Euler equations.

Bouchut, Brenier, Cortes, Ripoll, *JNS 2000*. F. Berthelin, P. Degond, M. Delitala, Rascle, *ARMA, 2008.* Degond, Navoret, Bon, Sanchez, *J. Stat. Phys., 2010*. Degond, Hua, Navoret, *J. Comput. Phys., 2011*. Berthelin, *M3AS, 2002, SIMA 2017*. Perrin, Zatorska, *Comm. PDEs 2015.* Perrin, Westdickenberg, *SIAM J. Math. Anal. 2018.*

1D dissipative Aw-Rascle system

Recall that taking $w_\varepsilon = u_\varepsilon + \partial_x p_\varepsilon(\rho_\varepsilon)$ we get (formally)

$$\partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0,$$
$$\partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho_\varepsilon u_\varepsilon^2) - \partial_x\left(\lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon\right) = 0,$$

where

$$\lambda_\varepsilon(\rho_\varepsilon) = \rho_\varepsilon^2 p_\varepsilon'(\rho_\varepsilon), \qquad p_\varepsilon(\rho_\varepsilon) = \varepsilon \frac{\rho_\varepsilon^\gamma}{(1-\rho_\varepsilon)^\beta}, \quad \gamma \geq 0, \quad \beta > 1.$$

What is the asymptotic limit of this model when $\varepsilon \to 0$?

A. Lefebvre-Lepot and B. Maury, Micro-Macro Modelling of an Array of Spheres Interacting Through Lubrication Forces. *Adv. in Math.Sci. Appl.*, 21(2): 535–557, 2011.

Heuristic limit passage

Let us rewrite the system once more...

$$\partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0,$$
$$\partial_t \left(\rho_\varepsilon u_\varepsilon + \partial_x \pi_\varepsilon\right) + \partial_x\Big(\left(\rho_\varepsilon u_\varepsilon + \partial_x \pi_\varepsilon\right) u_\varepsilon \Big) = 0,$$

where we denoted $\pi_\varepsilon'(\rho_\varepsilon) = \rho_\varepsilon p_\varepsilon'(\rho_\varepsilon)$.

From the a-priori estimates:

$$\rho_\varepsilon \rightharpoonup \rho \quad \text{weakly-* in} \quad L^\infty_{t,x}, \qquad u_\varepsilon \rightharpoonup u \quad \text{weakly-* in} \quad L^\infty_{t,x},$$
$$\rho_\varepsilon u_\varepsilon \rightharpoonup m \quad \text{weakly-* in} \quad L^\infty_{t,x}, \qquad \pi_\varepsilon(\rho_\varepsilon) \rightharpoonup \pi \quad \text{weakly in} \quad L^2_t H^1_x,$$

and also

$$(1-\rho_\varepsilon)\pi_\varepsilon(\rho_\varepsilon) \to 0 \quad \text{strongly in } L^q_{t,x}, \qquad (1-\rho_\varepsilon)\pi_\varepsilon(\rho_\varepsilon) \rightharpoonup (1-\rho)\pi \quad \text{in } \mathcal{D}'_{t,x}.$$

Hence, passing to the limit in the system, we verify that:

$$\partial_t \rho + \partial_x m = 0,$$
$$\partial_t(m + \partial_x \pi) + \partial_x\left(\overline{(m - \partial_x \pi)u}\right) = 0,$$
$$(1-\rho)\pi = 0.$$

Duality solutions of Bouchut and James, NA '98, Comm. PDEs '99.

Definition A solution $b \in \mathrm{Lip}_{loc}([0,T] \times \mathbb{T})$ to

$$\partial_t b + u_\varepsilon \partial_x b = 0, \qquad b_{|t=T} = b_T \tag{1}$$

is said to be reversible if there exist two solutions $b_1, b_2 \in \mathrm{Lip}_{loc}([0,T] \times \mathbb{T})$ of (1) such that $\partial_x b_1 \geq 0$, $\partial_x b_2 \geq 0$ and $b = b_1 - b_2$.

Remark Bouchut and James showed that the backward problem (1) is well-posed in the class of reversible solutions provided $u_\varepsilon \in L^\infty([0,T] \times \mathbb{T})$, and if u_ε satisfies the *Oleinik entropy condition*, i.e. $\partial_x u_\varepsilon \leq 1/t$.

Definition We say that $\mu \in \mathcal{C}([0,T], \mathcal{M}_{loc,x})$ is a duality solution to

$$\partial_t \mu + \partial_x(\mu u) = 0 \quad \text{in }]0,T[\times \mathbb{T}$$

if, for any $0 < \tau \leq T$, and any reversible solution b, the function

$$t \mapsto \int_{\mathbb{T}} b(t,x)\mu(t,dx)$$

is constant on $[0,\tau]$.

Application to the Aw-Rascle system asymptotic

We use the notion of duality solutions to show that for $\varepsilon \to 0$ the solutions of

$$\begin{aligned} &\partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0, \\ &\partial_t \underbrace{\left(\rho_\varepsilon u_\varepsilon + \partial_x \pi_\varepsilon\right)}_{:=q_\varepsilon} + \partial_x\Big(\underbrace{\left(\rho_\varepsilon u_\varepsilon + \partial_x \pi_\varepsilon\right)}_{:=q_\varepsilon} u_\varepsilon\Big) = 0, \end{aligned}$$

converge (in some sense) to the duality solution of

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0, \\ &\partial_t q + \partial_x\big(q u\big) = 0, \\ &0 \le \rho \le 1, \quad (1-\rho)\pi = 0, \quad \pi \ge 0, \end{aligned}$$

with the unknowns are ρ, u, π and where $q = \rho u + \partial_x \pi$.

L. Boudin, *SIMA*, 32(1): 172–193, 2000.

N. Chaudhuri, L. Navoret, C. Perrin, E. Zatorska, *coming soon*.

Thank you!