Random Matrices from the Classical Compact Groups: a Panorama

Part VI: Asymptotics for eigenvalue distributions

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Asymptotic regimes

The largest portion of Random Matrix Theory focuses on the distributions of eigenvalues when $n \to \infty$, in the

- macroscopic regime:
  - all the eigenvalues,
  - the whole circle S^1,
  - gaps $\approx \frac{1}{n}$,
- microscopic regime:
  - a fixed number the eigenvalues,
  - an arc of length $\approx \frac{1}{n}$,
  - gaps ≈ 1.
- mesoscopic regime: anything in between.

Classical limit theorems

The macroscopic limit theorems are analogous to the classical limit theorems of probability.

Let $\{X_i\}$ be i.i.d. random vectors in $\mathbb{R}^d$.

Law of large numbers:

$$\left\langle \frac{1}{n} \sum_{i=1}^{n} X_i, v \right\rangle \xrightarrow{n \to \infty} \langle \mathbb{E} X_1, v \rangle$$

a.s. for every $v \in \mathbb{R}^d$ (equivalently, all v in a basis).

Classical limit theorems

Central limit theorem:

$$\frac{\left\langle \frac{1}{n}\sum_{i=1}^{n} X_i, v \right\rangle - \langle \mathbb{E} X_1, v\rangle}{\sqrt{\mathrm{Var}\left\langle \frac{1}{n}\sum_{i=1}^{n} X_i, v \right\rangle}} \xrightarrow[D]{n\to\infty} N(0,1)$$

for every $v \in \mathbb{R}^d$.

Large deviations principle (Cramér's theorem):

$$\frac{1}{n}\log \mathbb{P}\left[\frac{1}{n}\sum_{i=1}^{n} X_i \in A\right] \xrightarrow{n\to\infty} -\inf_{x\in A} \Lambda^*_{X_1}(x)$$

for nice $A \subseteq \mathbb{R}^d$.

Empirical spectral measure

Macroscopic random matrix theory considers the empirical spectral measure of U:

$$\mu_U = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j}.$$

Classical limit theory	Random matrix theory
$\frac{1}{n}\sum_{j=1}^{n} X_j$	μ_U
$\mathbb{R}^d$	$M(S^1)$ or $P(S^1)$
$\left\langle \frac{1}{n}\sum_{j=1}^{n} X_j, v \right\rangle$	$\int f \, d\mu_U = \frac{1}{n}\sum_{j=1}^{n} f(\lambda_j)$ (linear eigenvalue statistic)

Expectations

ν denotes the uniform measure on S^1.

If $\mathbb{G} = \mathbb{U}(n)$ is random then $\mathbb{E}\mu_U = \nu$ by symmetry.

If G is one of the other groups then symmetry isn't enough, but the Diaconis–Shahshahani calculations imply

$$\mathbb{E} \int z^k \, d\mu_U \xrightarrow{n\to\infty} \delta_{k,0} = \int z^k \, d\nu$$

for all $k \in \mathbb{Z}$, so

$$\mathbb{E} \int f \, d\mu_U \xrightarrow{n\to\infty} \int f \, d\nu$$

for all nice f.

That is, $\mathbb{E}\mu_U \xrightarrow{n\to\infty} \nu$.

Law of large numbers

Theorem (Diaconis–Shahshahani)

For any nice f and any of the groups,

$$\int f \, d\mu_U \xrightarrow{n\to\infty} \int f \, d\nu$$

with probability one.

That is, $\mu_U \xrightarrow{n\to\infty} \nu$ weakly almost surely.

A few quick proofs:

- Compute $\mathrm{Var} \int z^k \, d\mu_U$ from Diaconis–Shahshahani, use Chebyshev and Borel–Cantelli.
- Measure concentration + Borell–Cantelli (as in Part III).
- $\mathbb{E}\mu_U(A) = \int \mathbb{1}_A \, d\mu_U$ and $\mathrm{Var}\, \mu_U(A)$ can be estimated using the determinantal kernel. (More on this next time.)

Central limit theorems

Theorem (Soshnikov)

For any $\mathbb{G}$, if $A \subseteq S^1$ is a fixed arc, then

$$\frac{N_A - \mathbb{E}N_A}{\sqrt{\operatorname{Var} N_A}} = \frac{\mu_U(A) - \mathbb{E}\mu_U(A)}{\sqrt{\operatorname{Var} \mu_A(A)}} \xrightarrow[D]{n\to\infty} N(0,1).$$

Update of Soshnikov's proof:

N_A is distributed as a sum of independent Bernoulli random variables.

$\operatorname{Var} N_A \approx \log n \to \infty$, so the Lindeberg central limit theorem applies.

Central limit theorems

Theorem (Wieand)

Let $U \in \mathbb{U}(n)$*. For* $0 \leq \alpha < \beta < 2\pi$*, define*

$$X_{\alpha,\beta} = \frac{\pi}{\sqrt{\log n}}\left(N_{[\alpha,\beta]} - \mathbb{E}N_{[\alpha,\beta]}\right).$$

Then any finite collection of $\{X_{\alpha,\beta}\}$ *converges in distribution as* $n \to \infty$ *to a centered jointly Gaussian family with*

$$\mathrm{Cov}(X_{\alpha,\beta}, X_{\alpha',\beta'}) = \begin{cases} 1 & \text{if } \alpha = \alpha', \beta = \beta', \\ 1/2 & \text{if } \alpha = \alpha', \beta \neq \beta', \\ 1/2 & \text{if } \alpha \neq \alpha', \beta = \beta', \\ -1/2 & \text{if } \alpha = \beta' \text{ or } \beta = \alpha', \\ 0 & \text{otherwise.} \end{cases}$$

Central limit theorems

Idea of Wieand's proof:

The multivariate moment generating function

$$\mathbb{E}e^{t_1 N_{A_1} + \cdots + t_k N_{A_k}} = \mathbb{E}\prod_{j=1}^{n} \exp\left(\sum_{i=1}^{k} t_i \mathbb{1}_{\lambda_j \in A_i}\right)$$

can be written as a Toeplitz determinant.

The surprising covariance structure has a simple explanation/interpretation...

Central limit theorems

Theorem (Diaconis–Evans)

Let $U \in \mathbb{U}(n)$. For f in the Bessel potential space (Sobolev space) $H^{1/2}(S^1)$, define

$$X_f = \int f \, d\mu_U - \int f \, d\nu.$$

Then any finite collection of $\{X_f\}$ converges in distribution as $n \to \infty$ to a centered jointly Gaussian family with

$$\mathrm{Cov}(X_f, X_g) = \langle f, g \rangle_{H^{1/2}} .$$

Central limit theorems

Idea of Diaconis–Evans's proof:
Use (refinements of) Diaconis–Shahshahani computations and Fourier approximation of $f \in H^{1/2}$.

Indicators of intervals are not in $H^{1/2}$, but the method can be extended to recover Soshnikov/Wieand's results.

The methods extend to $\mathbb{O}(n)$ and $\mathbb{Sp}(n)$.

Large deviations principle

Theorem (Hiai–Petz)

Let $U \in \mathbb{U}(n)$. For a nice $A \subseteq P(S^1)$,

$$\frac{1}{n^2} \log \mathbb{P}\left[\mu_U \in A\right] \xrightarrow{n\to\infty} - \inf_{\rho \in P(A)} \left[- \iint \log |z - w| \; d\rho(z)\, d\rho(w) \right]$$

(roughly).

The quantity in the inf is the logarithmic energy / free entropy $\mathcal{E}(\rho)$.

$\mathcal{E}(\rho) \geq 0$, with $=$ only for $\rho = \nu$.

Very roughly: $\mathbb{P}[\mu_U \in A] \approx e^{-n^2 \inf_A \mathcal{E}}$.

μ_U is very unlikely to be very different from ν.

Microscopic limits

The determinantal point process structure is at the heart of the microscopic regime.

The rescaled eigenangles $\{\frac{n}{2\pi}\theta_j\}$ of $U \in \mathbb{U}(n)$ are a DPP on $[-n/2, n/2]$ with kernel

$$\widetilde{K}_n(x,y) = \frac{\sin\left(\pi(x-y)\right)}{n\sin\left(\frac{\pi}{n}(x-y)\right)} \xrightarrow{n\to\infty} \frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)}.$$

Theorem

The point process of rescaled eigenangles $\{\frac{n}{2\pi}\theta_j\}$ of $U \in \mathbb{U}(n)$ converges as $n \to \infty$ to a DPP on $\mathbb{R}$ with kernel

$$K_{\text{sine}}(x,y) = \frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)}.$$

Microscopic limits

This almost immediately yields:

Corollary

The joint intensities of $\{\frac{n}{2\pi}\theta_j\}$ *converge as* $n \to \infty$ *to the joint intensities of the sine kernel process.*
The counting functions N_A *for the process* $\{\frac{n}{2\pi}\theta_j\}$ *converge in distribution as* $n \to \infty$ *to the counting functions of the sine kernel process.*

Microscopic limits

The DPP structure contains a lot of information about gaps/spacings as well, e.g.:

Proposition

The distribution of the gap between two successive points in a translation-invariant DPP on $\mathbb{R}$ has a density

$$\frac{d^2}{dx^2} \det(I - T_{(0,x)}),$$

where $T_{(0,x)}$ is the integral operator on $L^2(0,x)$ given by the DPP kernel and $\det$ is a Fredholm determinant.

Typical gaps

On average, the gap between adjacent eigenvalues is $\frac{2\pi}{n}$.

How does a random matrix tend to vary from that?

Theorem (Soshnikov)

For $s > 0$, let $\eta(s)$ be the number of gaps $\geq \frac{2\pi}{n} s$ between adjacent eigenvalues of $U \in \mathbb{U}(n)$.
Then

$$\frac{\eta(s) - \mathbb{E}\eta(s)}{\sqrt{\operatorname{Var} \eta(s)}} \xrightarrow[D]{n\to\infty} N(0,1).$$

This result extends in various ways:

- to mesoscopic scales,
- to a process-level result for $s > 0$,
- to other groups.

Small gaps

Theorem (Vinson, Ben Arous–Bourgade)

Let γ_k denote the k^{th} smallest gap between adjacent eigenvalues of $U \in \mathbb{U}(n)$.
Then $n^{4/3}\gamma_k$ converges in distribution as $n \to \infty$ to a random variable with density on $(0, \infty)$

$$\frac{3}{(k-1)!} x^{3k-1} e^{-x^3}.$$

Moreover, the point process $\{n^{4/3}\gamma_i\}$ converges to a Poisson point process with explicit intensity.

Big gaps

Theorem (Feng–Wei)

Let Γ_k denote the k^{th} largest gap between adjacent eigenvalues of $U \in \mathbb{U}(n)$.
Then

$$\widetilde{\Gamma}_k = \frac{\sqrt{\log n}}{2\sqrt{2}}\left(n\Gamma_k - \sqrt{32 \log n}\right) - \frac{3}{8}\log(2\log n)$$

converges in distribution as $n \to \infty$ to a Gumbel random variable with a certain mean.

Moreover, the point process $\{\widetilde{\Gamma}_i\}$ converges to a Poisson point process with explicit intensity.

In particular, $\Gamma_k \sim \frac{\sqrt{32 \log n}}{n}$.

What else?

Some other types of asymptotic spectral results:

- Mesoscopic results
- Asymptotics for characteristic polynomials.
- Eigenvalues of truncations of Haar-distributed random matrices.

Additional references

- Alexander Soshnikov, "Level spacings distribution for large random matrices: Gaussian fluctuations", *Ann. of Math. (2)* 148, pp. 573–617, 1998.
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- Gérard Ben Arous and Paul Bourgade, "Extreme gaps between eigenvalues of random matrices", *Ann. Probab.* 41, pp. 2648–2681, 2013.
- Renjie Feng and Dongyi Wei, "Large gaps of CUE and GUE", arXiv preprint arXiv:1807.02149, 2018.