Random Matrices from the Classical Compact Groups: a Panorama

Part VII: Nonasymptotic high-dimensional eigenvalue behavior

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Oxford, 8 March 2021

Nonasymptotic random matrix theory

Classical random matrix theory focuses on limits as $n \to \infty$ (Part VI).

Nonasymptotic random matrix theory refers to results (usually inequalities) involving quantities which are independent of n, e.g.

$$\mathbb{P}[\text{Event}(n)] \leq Ce^{-cn}.$$

This statement is trivial for small n, but very strong for large n.

Nonasymptotic results are crucial for applications to geometry, statistics, computer science, ... (Part IV).

Nonasymptotic RMT blends macroscopic/microscopic scales.

Eigenvalue counting functions

Recall that the eigenvalues of a random $U \in \mathbb{G}$ form a determinantal point process of n points with continuous Hermitian kernel K.

In general:

$$\mathbb{E}N_A = \int_A K(x,x)\, dx\, dx,$$

$$\mathrm{Var}\, N_A = \int_A \int_{A^c} |K(x,y)|^2\; dx\, dy.$$

For eigenvalues of $U \in \mathbb{G}$:

$$\left| \mathbb{E}N_A - \frac{n}{2\pi} |A| \right| < 1,$$

$$\mathrm{Var}\, N_A \leq C \log n.$$

Concentration of eigenvalues counts

Since N_A has the distribution of a sum of independent Bernoulli random variables, Bernstein's inequality implies

$$\mathbb{P}\left[\left|N_A - \frac{n}{2\pi}|A|\right| \geq t\right] \leq C \exp\left(-\frac{ct^2}{\log n + t}\right)$$

for all $t > 0$.

This implies a nonasymptotic rigidity for eigenvalues:

Proposition (E. Meckes and M.M.)

Let $0 \leq \theta_1 \leq \cdots \leq \theta_n < 2\pi$ be the eigenangles of $U \in \mathbb{G}$. Then for each j,

$$\mathbb{P}\left[\left|\theta_j - \frac{2\pi j}{n}\right| \geq \frac{t}{n}\right] \leq C \exp\left(-\frac{ct^2}{\log n + t}\right).$$

Comparison of eigenvalue counts

Theorem (E. Meckes and M.M.)

Let $A \subseteq \mathbb{R}$ be an interval, $U_n \in \mathbb{U}(n)$, and let $N_A^{(m)}$ be the number of eigenangles of U_{mn} in $\frac{1}{m}A$. Then

$$d_{TV}(N_A, N_A^{(m)}) \leq C\sqrt{mn}\,|A|^2 .$$

This is small as long as $|A| \ll n^{-1/4}$.

Corollary

$d_{TV}(N_{\frac{1}{n}A}, N_{\frac{1}{n}A}^{\text{sine}}) \leq Cn^{-3/2}$.

The proof of the theorem uses a general comparison principle for DPPs, based on couplings of independent Bernoulli random variables.

The spectral measure

$\mu_U = \frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_j}$ is the empirical spectral meaure of U.

ν is the uniform measure on S^1

The L^1-Wasserstein distance is

$$W_1(\mu_U,\nu) = \sup_{\|\psi\|_L \le 1}\left|\int \psi \, d\mu_U - \int \psi \, d\nu\right|,$$

where $\|\psi\|_L = \sup_{x \neq y}\frac{|\psi(x)-\psi(y)|}{\|x-y\|_2}$.

The Kolmogorov distance is

$$d_K(\mu_U,\nu) = \sup_A |\mu_U(A) - \nu(A)|,$$

where the sup is over arcs $A \subseteq S^1$.

The spectral measure

Theorem (E. Meckes and M.M.)

If $U \in \mathbb{G}$ is random then $\mathbb{E} W_1(\mu_U, \nu) \leq C \frac{\sqrt{\log n}}{n}$.

Idea of proof: Eigenvalue rigidity.

Theorem (E. Meckes and M.M.)

If $U \in \mathbb{U}(n)$ is random then $c \frac{\log n}{n} \leq \mathbb{E} d_K(\mu_U, \nu) \leq C \frac{\log n}{n}$.

Idea of proof for lower bound: Negative association for DPPs.

If $\mu = \frac{1}{n} \sum_{j=1}^{n} \delta_{X_j}$ for $\{X_j\}$ i.i.d. uniform in S^1, then

$$\mathbb{E} W_1(\mu, \nu) \approx \frac{1}{\sqrt{n}} \approx \mathbb{E} d_K(\mu, \nu).$$

Concentration for linear eigenvalue statistics

Let $f : S^1 \to \mathbb{R}$ be 1-Lipschitz.

Concentration of measure and the Hoffman–Wielandt inequality imply

$$\mathbb{P}\left[\left|\int f\, d\mu_U - \mathbb{E}\int f\, d\mu_U\right| \geq t\right] \leq 2e^{-cn^2t^2}.$$

This and the previous estimates imply

$$\mathbb{P}\left[\left|\int f\, d\mu_U - \int f\, d\nu\right| \geq t\right] \leq 2e^{-cn^2t^2}$$

for $t \gtrsim \frac{\sqrt{\log n}}{n}$.

Concentration for linear eigenvalue statistics

We have

$$\mathbb{P}\left[\left|\int f\, d\mu_U - \int f\, d\nu\right| \geq t\right] \leq 2e^{-cn^2t^2}$$

for every $t \gtrsim \frac{\sqrt{\log n}}{n}$ and every n.

The large deviations principle (Hiai–Petz) implies

$$\frac{1}{n^2}\log \mathbb{P}\left[\left|\int f\, d\mu_U - \int f\, d\nu\right| \geq t\right] \xrightarrow{n\to\infty} -\alpha(f,t)$$

for fixed t.

Concentration for traces of powers

The function $z \mapsto z^m$ is m-Lipschitz on S^1, and so for $U \in \mathbb{U}(n)$ we have

$$\mathbb{P}\left[\left|\operatorname{Tr} U^m - n\delta_{m,0}\right| \geq t\right] \leq 2e^{-ct^2/m^2}.$$

We can do better using the result of Rains: if $1 \leq m \leq n$,

$$\operatorname{Tr} U^m \sim \sum_{k=1}^{m} \operatorname{Tr} U_k$$

for $U_k \in \mathbb{U}(n/m)$ independent, and so

$$\mathbb{P}\left[\left|\operatorname{Tr} U^m - n\delta_{m,0}\right| \geq t\right] \leq 2e^{-ct^2/m}$$

(consistent with Diaconis–Shahshahani).

Concentration of the spectral measure

μ_U is itself a Lipschitz function of U w.r.t. W_1.

Theorem (E. Meckes and M.M.)

$$\mathbb{P}\left[W_1(\mu_U, \nu) \geq C\frac{\sqrt{\log n}}{n} + t\right] \leq e^{-cn^2t^2}.$$

Thus with probability 1, $W_1(\mu_U, \nu) \leq C\frac{\sqrt{\log n}}{n}$ for all sufficiently large n.

The crucial matrix-analytic property is that U is normal.

Spectra of powers

The proofs can be combined with Rains's theorem:

Theorem (E. Meckes and M.M.)

For each $1 \leq m \leq n$,

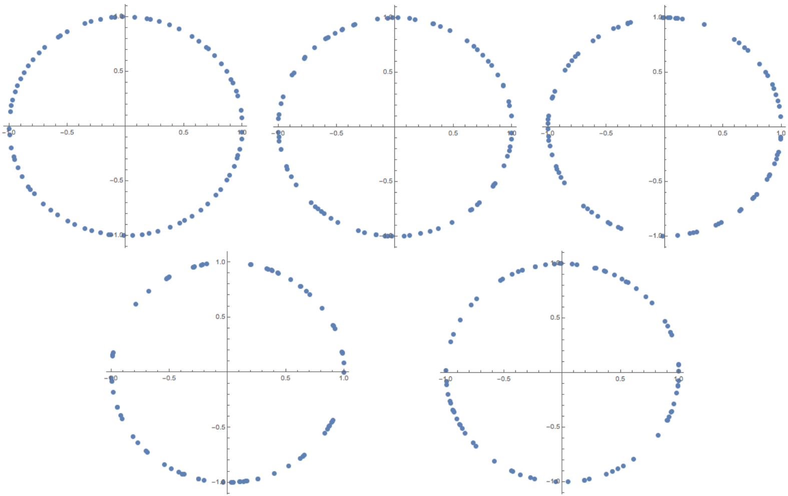
$$\mathbb{P}\left[W_1(\mu_{U^m}, \nu) \geq C\frac{\sqrt{m(\log(n/m)+1)}}{n} + t\right] \leq e^{-cn^2t^2/m}.$$

If $m = m(n)$*, then with probability* 1,

$$W_1(\mu_{U^m}, \nu) \leq C\frac{\sqrt{m(\log(n/m)+1)}}{n}$$ *for all sufficiently large* n.

Thus μ_{U^m} smoothly interpolates between the behavior of μ_U and of i.i.d. samples.

Spectra of powers



Eigenvalues of U^m for $U \in \mathbb{U}(80)$ and $m = 1, 5, 20, 45, 80$.

Rates of convergence in CLTs

Theorem (Döbler–Stolz)

For $U \in \mathbb{U}(n)$ and $d \leq n/2$, let

$$X = (\operatorname{Tr} U, \operatorname{Tr} U^2, \ldots, \operatorname{Tr} U^d)$$

and

$$Y = (Z_1, \sqrt{2} Z_2, \ldots, \sqrt{d} Z_d),$$

where $\{Z_j\}$ are i.i.d. standard complex normals.

Then $W_1(X, Y) \leq C \dfrac{d^{7/2}}{n}$.

Idea of proof: Diaconis–Shahshahani plus Stein's method.

Rates of convergence in CLTs

Corollary (Döbler–Stolz)

For $U \in \mathbb{U}(n)$ and smooth $f : S^1 \to \mathbb{R}$,

$$W_1\left(n\left(\int f\, d\mu_U - \int f\, d\nu\right), N\left(0, \|f\|^2_{H^{1/2}}\right)\right) = O\left(\frac{1}{n^{1-\varepsilon}}\right)$$

for every $\varepsilon > 0$.

Idea of proof: Last result plus Fourier approximation.

It is crucial here to be able to let d grow with n.

Rates of convergence in CLTs

Theorem (Johansson)

Let $U \in \mathbb{U}(n)$ and let Z be a standard complex normal random variable. Then

$$d_{TV}\left(\frac{1}{\sqrt{k}}\operatorname{Tr} U^k, Z\right) \leq Ce^{-c_k n \log n}.$$

Slightly weaker versions hold for the other groups.

Multivariate versions have been proved very recently by Johansson–Lambert ($\mathbb{U}(n)$) and Courteaut–Johansson (other groups).

A question of Diaconis

In a talk in memory of Elizabeth Meckes, Persi Diaconis observed:

> *Diaconis–Shahshahani and Johannson show that* $\operatorname{Tr} U^k$ *is remarkably similar in distribution to* $\sqrt{k}Z$.
>
> *In particular, there must be a coupling of these random variables in which they are nearly equal.*

And asked:

> *Can we construct such a coupling?*

Additional references

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- Kurt Johansson and Gaultier Lambert, "Multivariate normal approximation for traces of random unitary matrices".
- Klara Courteaut and Kurt Johansson, "Multivariate normal approximation for traces of orthogonal and symplectic matrices".
- Persi Diaconis, "Haar-distributed random matrices — in memory of Elizabeth Meckes" (talk).