Uniqueness and Stability for Shock Reflection problem

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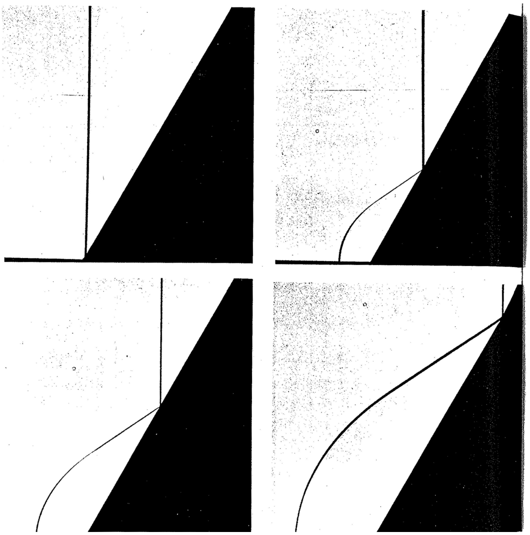
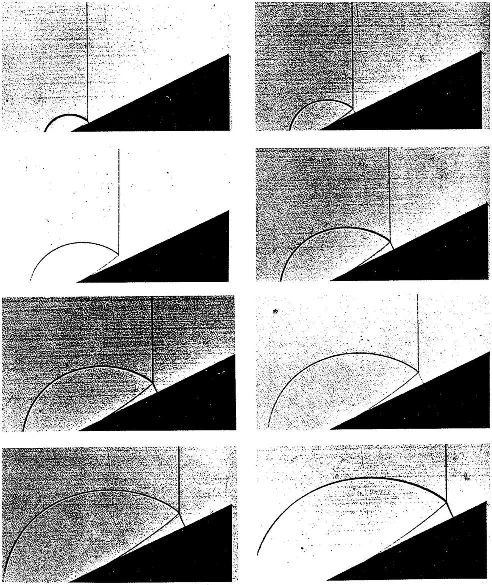
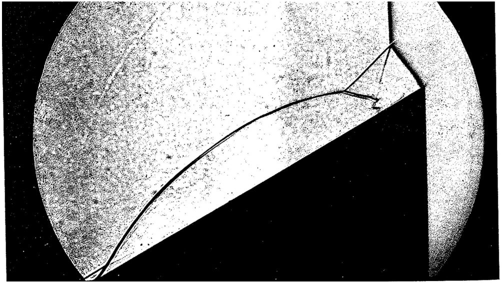
Shock reflection by a wedge: Regular reflection



Shock reflection by a wedge: Mach reflection



Shock reflection by a wedge: Irregular Mach reflection.



Self-similar flow: $(\vec{u}, p, \rho)(x, t) = (\vec{u}, p, \rho)(\frac{x}{t})$.

Shock reflection

First described by E. Mach 1878. Reflection patterns: Regular reflection, Mach reflection.

J. von Neumann, 1940s: on transition between patterns

Later works: experimental, computational. Asymptotic analysis: Lighthill, Keller, Blank, Hunter, Harabetian, Morawetz.
Reference: book by J. Glimm and A. Majda, survey by D. Serre.

Analysis: Special models (Transonic small disturbance eq., pressure-gradient system, nonlinear wave eq.): Gamba, Rosales, Tabak, Canic, Keyfitz, Kim, Lieberman, Y.Zheng, G.-Q. Chen-W. Xiang.

Local existence results: S.-X. Chen.

More recent results for potential flow:

Existence of global shock reflection solutions for potential flow: G.-Q.Chen-F., Elling

The complete up-to-date results on existence of regular reflection solutions and their proofs are presented in the monograph "The Mathematics of Shock Reflection-Diffraction and von Neumann conjectures" by G.-Q.Chen-F, 2018.

Other self-similar shock reflection problems:

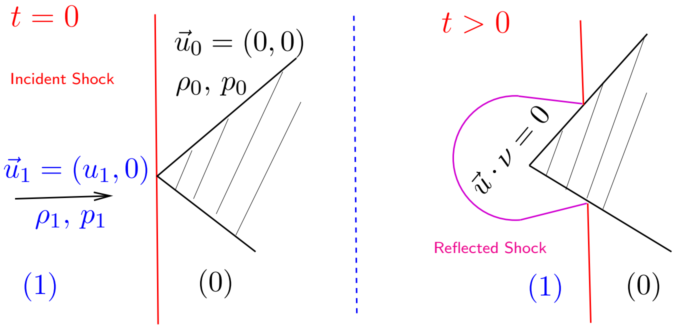
Prandtl Reflection: Elling-Liu, Bae-G.-Q.Chen-F

Shock interactions/reflection for Chaplygin gas: D. Serre

Properties of solutions of self-similar reflection problems: Bae-G.-Q.Chen-F, G.-Q. Chen-F.-W. Xiang, Elling.

Stability and uniqueness of regular reflection solutions. G.-Q. Chen-F.-W. Xiang.

Shock reflection as a Riemann problem



Initial data: Constant (uniform) states (0) and (1):
State (0): velocity $\vec{u}_0 = (0, 0)$, density ρ_0, pressure p_0.
State (1): velocity $\vec{u}_1 = (u_1, 0)$, density ρ_1, pressure p_1.

$t > 0$: Self-similar solution of compressible Euler system:
$(\vec{u}, \rho, p) = (\vec{u}, \rho, p)(\vec{\xi})$, where $\vec{\xi} = \dfrac{\vec{x}}{t}$.

Potential flow system

Conservation of mass, Bernoulli's law

$$\rho_t + \textbf{div}(\rho\,\nabla\Phi) = 0,$$
$$\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + \frac{\rho^{\gamma-1}-1}{\gamma-1} = const$$

where:

$\vec{u} = (u_1, u_2)$ – velocity

Φ – velocity potential: $\vec{u} = \nabla_x\Phi$.

ρ – density

$p = \rho^\gamma$ – pressure

$\gamma > 1$ – adiabatic exponent (it is a given constant)

Compressible Euler system: Isentropic case

$$\partial_t\rho + \textbf{div}(\rho\,\vec{u}) = 0,$$
$$\partial_t(\rho\vec{u}) + \textbf{div}(\rho\,\vec{u}\otimes\vec{u}) + \nabla p = 0$$

Uniqueness/nonuniqueness for 2-D Riemann problems in whole space

Riemann problem in whole space for Euler system:

Chiodaroli-DeLellis-Kreml(2015): 2D isentropic Euler system
1) Entropy solutions of Riemann problem are non-unique in the class of entropy solutions isentropic Euler system.

2) Self-similar solutions of 1D structure in 2D with flat shock are unique (reduced to 1D system of conservation laws).

Non-uniqueness results for 2D full Euler system: S. Markfelder and C. Klingenberg (2017), Al Baba, Klingenberg, Kreml, Macha, Markfelder (2019)

Uniqueness/nonuniqueness for Shock Reflection as Multi-D Riemann problem in domain with boundary

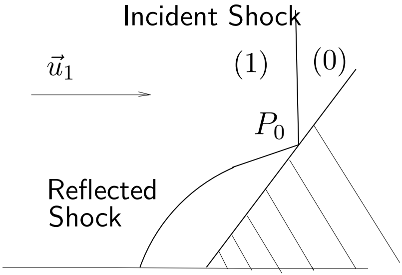
Uniqueness for shock reflection can be considered in class of:

1. Time-dependent solutions for compressible Euler system: non-uniqueness for normal reflection, using technique of Chiodaroli-DeLellis-Kreml;

2. Potential flow: uniqueness/nonuniqueness of general self-similar solutions (???);

3. Potential flow: Uniqueness of regular reflection solutions with convex shocks (G.-Q. Chen - F.-W. Xiang). Existence of "admissible" regular reflection solutions: G.-Q. Chen - F.; convexity of shocks for "admissible solutions": G.-Q. Chen - F.-W. Xiang.

Regular reflection in self-similar coordinates $\vec{\xi} = \frac{\vec{x}}{t}$



Given:
State (0): velocity $\vec{u}_0 = (0, 0)$, density ρ_0, pressure p_0.
State (1): velocity $\vec{u}_1 = (u_1, 0)$, density ρ_1, pressure p_1.

Problem: Find self-similar solution: $(\vec{u}, \rho, p) = (\vec{u}, \rho, p)(\vec{\xi})$, where $\vec{\xi} = \dfrac{\vec{x}}{t}$, with asymptotic conditions at infinity determined by states (0) and (1), and satisfying $u \cdot \nu = 0$ on the boundary.

Potential flow: self-similar case

$\Phi(\vec{x},t) = t\psi(\xi,\eta)$, $\rho(\vec{x},t) = \rho(\xi,\eta)$ with $(\xi,\eta) = \frac{\vec{x}}{t} \in \mathbb{R}^2$.

Pseudo-potential: $\varphi = \psi - \frac{1}{2}(\xi^2 + \eta^2)$.

Equation for φ:

$$\text{div}\,\big(\rho(|\nabla\varphi|^2,\varphi)\nabla\varphi\big) + 2\rho(|\nabla\varphi|^2,\varphi) = 0,$$

with $$\rho(|\nabla\varphi|^2,\varphi) = \big(K - (\gamma-1)(\varphi + \frac{1}{2}|\nabla\varphi|^2)\big)^{\frac{1}{\gamma-1}}.$$

Equation is of mixed type:

elliptic $|\nabla\varphi| < c(|\nabla\varphi|^2,\varphi,K)$,

hyperbolic $|\nabla\varphi| > c(|\nabla\varphi|^2,\varphi,K)$,

where **sonic speed** c is:

$$c^2 = \rho^{\gamma-1} = K - (\gamma-1)(\varphi + \frac{1}{2}|\nabla\varphi|^2).$$

Uniform states

Solutions with constant (physical) velocity (u, v):

$$\varphi(\xi, \eta) = -\frac{\xi^2 + \eta^2}{2} + u\xi + v\eta + const.$$

Any such function is a solution.
Also (from formula) density $\rho(\nabla\varphi, \varphi) = const$, thus sonic speed $c = \rho^{\frac{\gamma-1}{2}} = const$. Then ellipticity region

$$|\nabla\varphi(\xi, \eta)| = |(u, v) - (\xi, \eta)| < c$$

is circle, centered at (u, v), radius c.

Shocks, RH conditions, Entropy condition

Shocks are discontinuities in the pseudo-velocity $\nabla\varphi$:

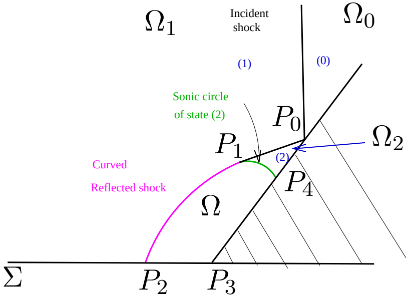
if Ω^+ and $\Omega^- := \Omega \setminus \overline{\Omega^+}$ are nonempty and open, and $S := \partial\Omega^+ \cap \Omega$ is a C^1 curve where $\nabla\varphi$ has a jump, then $\varphi \in C^1(\Omega^\pm \cup S) \cap C^2(\Omega^\pm)$ is a global weak solution in Ω if and only if φ satisfies potential flow equation in $\Omega^\pm$ and the Rankine-Hugoniot (RH) condition on S:

$$[\varphi]_S = 0,$$
$$\left[\rho(|\nabla\varphi|^2, \varphi)\nabla\varphi \cdot \nu\right]_S = 0,$$

where $[\cdot]_S$ is jump across S.

Entropy Condition on S: density increases across S in the flow direction.

Shock reflection as a free boundary problem



$$\text{div}\left(\rho(|\nabla\varphi|^2,\varphi)\nabla\varphi\right) + 2\rho(|\nabla\varphi|^2,\varphi) = 0 \quad \text{in } \Omega,$$

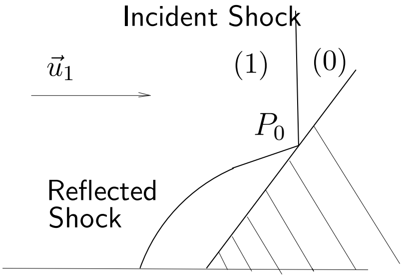
$$\left.\begin{array}{l} \rho(|\nabla\varphi|^2,\varphi)\nabla\varphi\cdot\nu = \rho(|\nabla\varphi_1|^2,\varphi_1)\nabla\varphi_1\cdot\nu \\ \varphi = \varphi_1 \end{array}\right\} \text{ on } P_1P_2$$

$\varphi = \varphi_2$ on P_1P_4 (and prove $D_\nu\varphi = D_\nu\varphi_2$ on P_1P_4)

$\varphi_\nu = 0$ on Wedge P_3P_4, Symmetry line P_2P_3,

Solve for: Free boundary P_1P_2 and function φ in Ω.
Expect equation elliptic in Ω.

Regular reflection, state (2)



φ = pseudo-potential between the reflected shock and the wall
φ_1 = pseudo-potential of state (1)

Denote $\nabla\phi(P_0) = (u_2, v_2)$, where $\phi = \varphi + \frac{\xi^2+\eta^2}{2}$. Since $\varphi_\nu = 0$ on wedge, then $v_2 = u_2 \tan\theta_w$.

Rankine-Hugoniot conditions at reflection point P_0, for φ and φ_1: algebraic equations for u_2, $\varphi(P_0)$

Regular reflection, state (2), detachment angle

If solution exists: Let

$$\varphi_2(\xi, \eta) = -(\xi^2 + \eta^2)/2 + u_2\xi + v_2\eta + C,$$

where C determined by $\varphi_2(P_0) = \varphi_1(P_0)$.

Existence of state (2) is necessary condition for existence of regular reflection

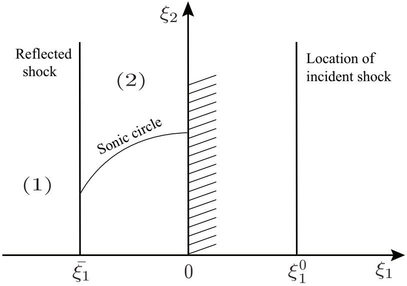
Given γ, ρ_0, ρ_1, there exists $\theta_{detach} \in (0, \frac{\pi}{2})$ such that:

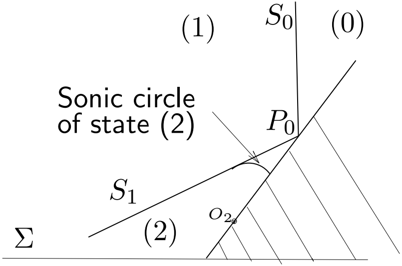
state (2) exists for $\theta_w \in (\theta_{detach}, \frac{\pi}{2})$,
state (2) does not exist for $\theta_w \in (0, \theta_{detach})$.

If φ_2 exist, then RH is satisfied along the line
$S_1 := \{\varphi_1 = \varphi_2\}$.

Weak and Strong State (2); Sonic angle

For each $\theta_w \in (\theta_{detach}, \frac{\pi}{2})$ there exists two possible States (2): weak and strong, with $\rho_2^{weak} < \rho_2^{strong}$. We always choose weak state (2). For strong state (2), existence of global regular reflection solution is not expected, Elling (2011) confirms that.





There exist $\theta_{sonic} \in (\theta_{detach}, \frac{\pi}{2})$ such that:

State 2 is supersonic at P_0 for $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$.

State 2 is subsonic at P_0 for $\theta_w \in (\theta_{detach}, \theta_{sonic})$.

Von Neumann's conjectures on transition between different reflection patterns

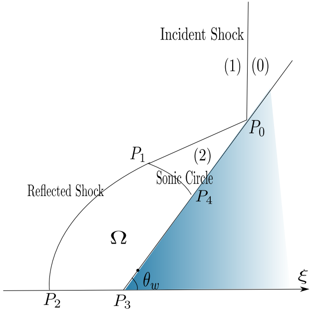
Recall: sonic angle θ_{sonic} and detachment angle θ_{detach} satisfy $0 < \theta_{detach} < \theta_{sonic} < \frac{\pi}{2}$.

Sonic conjecture:
Regular reflection for $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$, Mach reflection for $\theta_w \in (0, \theta_{sonic})$.

Von Neumann's detachment conjecture:
Regular reflection for $\theta_w \in (\theta_{detach}, \frac{\pi}{2})$, Mach reflection for $\theta_w \in (0, \theta_{detach})$.

G.-Q. Chen - F.(2018): existence of regular reflection for $\theta_w \in (\theta_{detach}, \frac{\pi}{2})$ for potential flow equation.
Structure of solutions: supersonic and subsonic regular reflections.

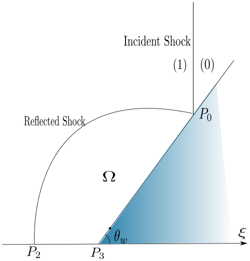
Supersonic regular reflection



Supersonic regular reflection: State (2) is supersonic at P_0.
Structure of solution φ:

- $\varphi = \varphi_i$ in Ω_i, i=0,1,2.
- $\varphi \in C^1(\overline{P_0P_2P_3})$, in particular C^1 across sonic arc P_1P_4.
- Shock P_0P_2 has flat part P_0P_1, curved part P_1P_2, and is C^1 across P_1.
- Equation is strictly elliptic in $\overline{\Omega} \setminus \overline{P_1P_4}$.

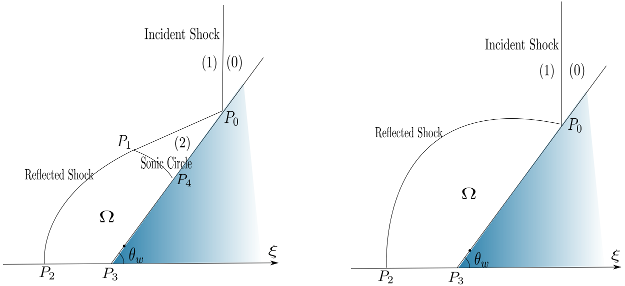
Subsonic regular reflection



Subsonic regular reflection: State (2) is subsonic at P_0.
Structure of solution φ:

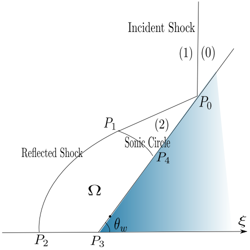
- $\varphi = \varphi_i$ in Ω_i, i=0,1.
- $\varphi \in C^1(\overline{P_0P_2P_3})$.
- $\varphi = \varphi_2$, $D\varphi = D\varphi_2$ at P_0.
- Shock P_0P_2 is C^1.
- Equation is strictly elliptic in $\overline{\Omega} \setminus \{P_0\}$.

Existence of regular reflection solutions



Theorem 1. (G.-Q. Chen-F.). If $\rho_1 > \rho_0 > 0$, $\gamma > 1$ then a regular reflection solution φ exists for all wedge angles $\theta_w \in (\theta_{detach}, \frac{\pi}{2})$. Here I skip some details related to "attached shocks" with $P_2 = P_3$. The type of reflection (supersonic or subsonic) for each θ_w is determined by the type of State 2 at the reflection point P_0 for θ_w. Moreover, solution satisfies the following additional properties:

Properties of solution: supersonic case



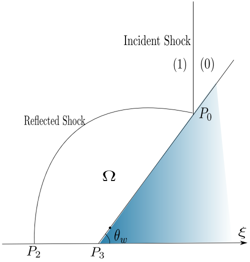
1) Equation is elliptic for φ in Ω, ellipticity degenerates near sonic arc $P_1 P_4$.

2) φ is $C^{1,1}$ near and across the sonic arc $P_1 P_4$;

3) Reflected shock is $C^{2,\beta}$, and a graph for a cone of directions $Con(\vec{e}_\eta, \vec{e}_{S_1})$ between $\vec{e}_\eta = (0,1)$ and $\vec{e}_{S_1} = P_0 P_1$;

4) $\varphi_2 \leq \varphi \leq \varphi_1$ in Ω, and $\partial_e(\varphi_1 - \varphi) < 0$ if $e \in Con(\vec{e}_\eta, \vec{e}_{S_1})$.

Properties of solution: subsonic case



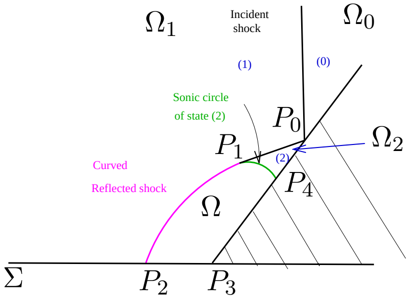
1) Equation is elliptic for φ in Ω, except for the sonic wedge angle (then ellipticity degenerates at P_0).

2) φ is $C^{2,\alpha}$ inside Ω, and $C^{1,\alpha}$ near and up to the reflection point P_0, and $\varphi = \varphi_2$, $D\varphi = D\varphi_2$ at P_0;

3) Reflected shock is $C^{2,\alpha}$ away from P_0 and $C^{1,\alpha}$ up to P_0, and a graph for a cone of directions $Con(\vec{e}_\eta, \vec{e}_{S_1})$;

4) $\varphi_2 \leq \varphi \leq \varphi_1$ in Ω, and $\partial_e(\varphi_1 - \varphi) < 0$ if $e \in Con(\vec{e}_\eta, \vec{e}_{S_1})$.

Stability of normal reflection as $\theta_w \to \pi/2$



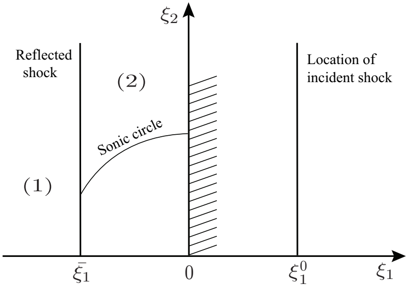
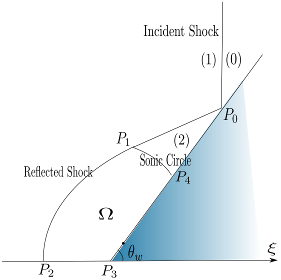
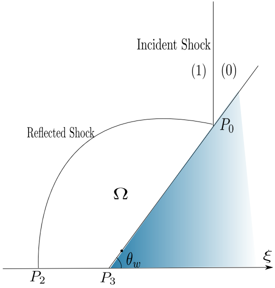


Figure: Normal reflection

Furthermore,
the solutions φ converge in $W^{1,1}_{loc}$ to the solution of the normal refection as $\theta_w \to \pi/2$.

Shock reflection: free boundary problem





$$\operatorname{div}\left(\rho(|\nabla\varphi|^2,\varphi)\nabla\varphi\right) + 2\rho(|\nabla\varphi|^2,\varphi) = 0 \ \text{ in } \Omega,$$

$$\left.\begin{array}{l} \rho(|\nabla\varphi|^2,\varphi)\nabla\varphi\cdot\nu = \rho(|\nabla\varphi_1|^2,\varphi_1)\nabla\varphi_1\cdot\nu \\ \varphi = \varphi_1 \end{array}\right\} \text{ on } P_1P_2$$

$\varphi = \varphi_2$ on P_1P_4 (and prove $D_\nu\varphi = D_\nu\varphi_2$ on P_1P_4)

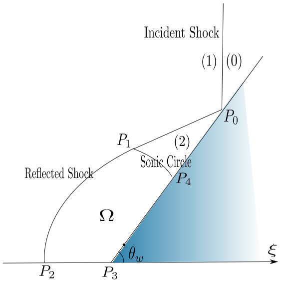
$\varphi_\nu = 0$ on Wedge P_3P_4, Symmetry line P_2P_3,

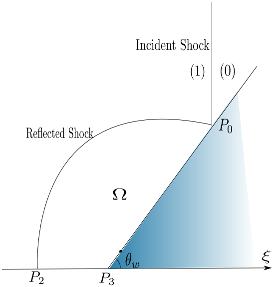
For subsonic reflection: $\varphi = \varphi_2$ and $D\varphi = D\varphi_2$ at P_0.

Solve for: Free boundary P_1P_2 (resp. P_0P_2 for subsonic case) and function φ in Ω. Expect equation elliptic in Ω.

Proof of Th. 1 is obtained by solving free boundary problem

using method of continuity/degree theory in the set of "admissible solutions"





Admissible solutions:

(a) Have structure supersonic or subsonic reflections depending on θ_w. Recall: this includes ellipticity in Ω and some regularity of P_0P_2 and of φ in $\overline{P_0P_2P_3}$;

(b) $\varphi_2 \leq \varphi \leq \varphi_1$ in Ω;

(c) satisfy nonstrict monotonicity $\partial_e(\varphi_1 - \varphi) \leq 0$ in Ω for any $e \in Con(e_\eta, e_{S_1})$.

Convexity of shock, uniqueness

Theorem 2. (Chen-F.-W. Xiang) For admissible solutions, shock is strictly convex in its relative interior.
Moreover, regular reflection solution satisfying (1)-(2) have cone of monotonicity (3) if and only if the shock is (strictly) convex.

Based on Theorem 2, we prove:

Theorem 3. (Chen-F.-Xiang) Admissible solutions are unique (and exist, by Thm. 1).

Corollary. (Chen-F.-Xiang) Regular reflections solutions with convex shocks are unique (and exist by Thms. 1, 2).

Outline of proof of uniqueness

By Th. 1, when $\theta_w \to \frac{\pi}{2}-$, admissible solutions converge to normal reflection. Also we have uniform estimates for admissible solutions. Then use the method of continuity:

Suppose φ, $\hat{\varphi}$ are two admissible solutions for some $\theta_w^* \in (\theta_w^d, \frac{\pi}{2})$. Then it is sufficient to:

1. Construct continuous in C^1 families $\theta_w \mapsto \varphi^{(\theta_w)}$, $\theta_w \mapsto \hat{\varphi}^{(\theta_w)}$ for $\theta_w \in [\theta_w^*, \frac{\pi}{2})$, with $\varphi^{(\theta_w^*)} = \varphi$, $\hat{\varphi}^{(\theta_w^*)} = \hat{\varphi}$,
2. Show "local uniqueness": if two admissible solutions for same θ_w are close in C^1, then they are equal.

Both are achieved if we can linearize FBP at an admissible solution, and linearization is "good" so that we can construct solutions for close wedge angles by Implicit Function Theorem.

Outline of proof of uniqueness

Rigorously, cannot use linearization for supersonic reflections: elliptic degeneracy near sonic arc requires very detail control of $D^2\varphi$ on sonic arc P_1P_4 to show well-posedness of linearization. We do not have this control at one point: P_1, where shock meets sonic arc.

Then we use a "nonlinear version of linearization": apply degree theory with "small" iteration set, consisting of functions close to the background solution (in appropriate norm). To apply degree theory, we need to show (in particular) that fixed point of iteration map cannot occur on the boundary of the iteration set. This is done using local uniqueness theorem.

We use convexity of shock for proof of local uniqueness theorem.

Proof of uniqueness: Role of convexity (heuristic)

When formally linearize FBP, variations of shock locations introduce an additional zero-order term in the oblique boundary condition derived from RH condition $\rho D\varphi \cdot \nu = \rho_1 D\varphi_1 \cdot \nu$. This term has the "correct" sign if shock is convex:

Formal linerization of RH conditions: shock is $\eta = f(\xi)$ with $\Omega \subset \{\eta < f(\xi)\}$ after rotating coordinates. Then RH:

$\varphi^\varepsilon(\xi, f^\varepsilon(\xi)) = \varphi_1(\xi, f^\varepsilon(\xi))$;

$\Big((\rho(|D\varphi^\varepsilon|^2, \varphi^\varepsilon)D\varphi^\varepsilon - \rho_1 D\varphi_1) \cdot (D\varphi_1 - D\varphi^\varepsilon)\Big)(\xi, f^\varepsilon(\xi)) = 0,$

where we use that $\nu = \frac{D\varphi_1 - D\varphi^\varepsilon}{|D\varphi_1 - D\varphi^\varepsilon|}$. Here $\varphi^\varepsilon = \varphi + \varepsilon\delta\varphi + \dots,$ same for f^ε. Taking $\frac{d}{d\varepsilon}$ at $\varepsilon = 0$ in 1st condition and using $\partial_\nu(\varphi_1 - \varphi) > 0$ and on shock, so $\partial_\eta(\varphi_1 - \varphi) > 0$:

$$\delta f = \frac{1}{\partial_\eta(\varphi_1 - \varphi)}\delta\varphi.$$

Now take $\frac{d}{d\varepsilon}$ at $\varepsilon = 0$ in 2nd RH condition

$$\Big((\rho(|D\varphi^\varepsilon|^2, \varphi^\varepsilon)D\varphi^\varepsilon - \rho_1 D\varphi_1) \cdot (D\varphi_1 - D\varphi^\varepsilon)\Big)(\xi, f^\varepsilon(\xi)) = 0,$$

Get two terms. First, linearization of oblique condition:

$$\frac{d}{d\varepsilon}\Big[\Big((\rho(|D\varphi^\varepsilon|^2, \varphi^\varepsilon)D\varphi^\varepsilon - \rho_1 D\varphi_1) \cdot (D\varphi_1 - D\varphi^\varepsilon)\Big)\Big]_{\varepsilon=0}(\xi, f(\xi))$$
$$= a\partial_\nu \delta\varphi + b\partial_\tau \delta\varphi + c\delta\varphi, \quad \text{where} \quad a(\xi) \geq \lambda > 0, \;\; c(\xi) \leq -\lambda < 0$$

Second term comes from the perturbation of shock location:

$$\partial_\eta\Big[\Big((\rho(|D\varphi|^2, \varphi)D\varphi - \rho_1 D\varphi_1) \cdot (D\varphi_1 - D\varphi)\Big)\Big]\delta f$$
$$= A(\varphi_1 - \varphi)_{\tau\tau}\delta f = \frac{A}{(\varphi_1 - \varphi)_\eta}(\varphi_1 - \varphi)_{\tau\tau}\delta\varphi,$$

where $A > 0$. Convexity of shock is equivalent to $(\varphi_1 - \varphi)_{\tau\tau} < 0$, and then the coefficient of $\delta\varphi$ has "correct" sign.

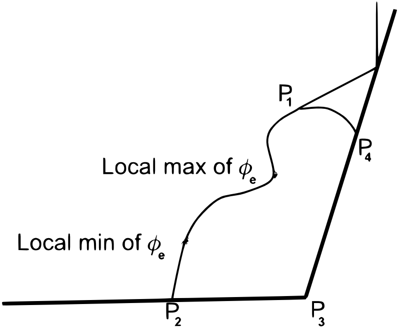
Outline of proof of convexity

Function $\phi = \varphi - \varphi_1$ satisfies equation

$$(c^2 - \varphi_\xi^2)\phi_{\xi\xi} - 2\varphi_\xi\varphi_\eta\phi_{\xi\eta} + (c^2 - \varphi_\eta^2)\phi_{\eta\eta} = 0,$$

where $c = c(|D\varphi|^2, \varphi)$ is the speed of sound, $c^2 = \rho^{\gamma-1}$. Equation is elliptic in Ω. $\phi = 0$ on $\Gamma_{shock} = P_1P_2$ (resp. on P_0P_2 for subsonic reflections). Also, $\phi < 0$ in Ω, which means $\phi_{\tau\tau} > 0$ on "strictly convex" parts of shock, and $\phi_{\tau\tau} < 0$ on parts of shock which are strictly convex in opposite direction.

Let $\boldsymbol{e} \in \mathbb{R}^2$, $\boldsymbol{e} \neq 0$. Then $v = \phi_{\boldsymbol{e}}$ satisfies equation $Lv = 0$ in Ω, where L is a linear elliptic 2nd order operator without zero order terms. From this and Rankine-Hugoniot conditions obtain, using maximum principles and Hopf's lemma:



Property 1

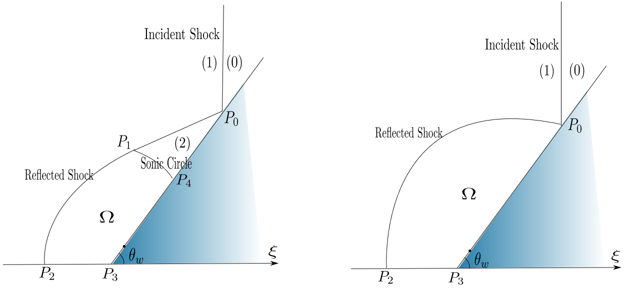
For $\boldsymbol{e} \in \mathbb{R}^2$ such that $\boldsymbol{e} \cdot \boldsymbol{\nu}_{sh} < 0$ on Γ_{shock}, where $\boldsymbol{\nu}_{sh}$ is interior unit normal:

If $\phi_{\boldsymbol{e}}$ has a local minimum relative to Ω at $P \in \Gamma_{shock}$, then $\phi_{\tau\tau}(P) > 0$.

If $\phi_{\boldsymbol{e}}$ has a local maximum relative to Ω at $P \in \Gamma_{shock}$, then $\phi_{\tau\tau}(P) < 0$.

Condition $\boldsymbol{e} \cdot \boldsymbol{\nu}_{sh} < 0$ on Γ_{shock} holds for any $\boldsymbol{e} \in Con(\vec{e}_{\eta}, \vec{e}_{S_1})$

We choose and fix $\boldsymbol{e} = \boldsymbol{\nu}_w$, where $\boldsymbol{\nu}_w$ is the interior (for Ω) unit normal on $\Gamma_{wedge} = P_3P_4$ (resp. $\Gamma_{wedge} = P_0P_3$ in the subsonic case). It satisfies: $\boldsymbol{\nu}_w \in Con(\vec{e}_\eta, \vec{e}_{S_1})$.



Function $v = \phi_{\boldsymbol{\nu}_w}$ satisfies oblique derivative condition on $\Gamma_{sym} = P_2P_3$; $\partial_{\boldsymbol{\nu}}(\varphi - \varphi_2) = 0$ on Γ_{wedge}, and $D(\varphi - \varphi_2) = 0$ on $\Gamma_{sonic} = P_1P_4$ (resp. at P_0 in the subsonic case). Also $\phi_{\boldsymbol{e}}$ is not constant in Ω.

From this, using that $\partial_{\boldsymbol{\nu}_w}(\varphi - \varphi_2) \leq 0$ in Ω, obtain: $\phi_{\boldsymbol{\nu}_w}$ cannot attain its local minimum (relative to Ω) on $\partial\Omega \setminus (\Gamma^0_{shock} \cup \{P_2\})$.

Convexity of Γ_{shock} is proved by a non-local argument.

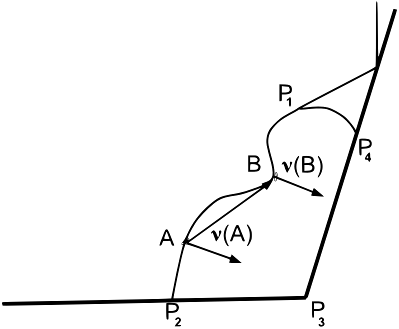
Technical tool: minimal (resp. maximal) chains.

Minimal chain $\{B_r(C^i)\}_{i=0}^k$ of (small) radius $r > 0$ is:

1) $C_0 \in \overline{\Omega}$
2) $C^{i+1} \in \overline{B_r(C^i) \cap \Omega}$ with $\phi_{\boldsymbol{\nu}_w}(C^{i+1}) = \min_{\overline{B_r(C^i)\cap\Omega}} \phi_{\boldsymbol{\nu}_w}$ for $i = 1, \ldots, k$.
3) Endpoint: $\phi_{\boldsymbol{\nu}_w}(C^k) = \min_{\overline{B_r(C^k)\cap\Omega}} \phi_{\boldsymbol{\nu}_w}$.

For any $C_0 \in \overline{\Omega}$ which is not a local minimum (resp. maximum) and small $r > 0$, minimal (resp. maximal) chain exists (for some finite $k \geq 1$), and $\cup_{i=0}^k B_r(C^i) \cap \Omega$ is connected using that angles are $< \pi$ at corners of Ω. Also, for sufficiently small r depending on various parameters, minimal/maximal chains do not intersect, using regularity $\|\phi\|_{C^{1,\alpha}(\overline{\Omega})} \leq C$.

Endpoint C^k is minimum (resp. maximum) of $\phi_{\boldsymbol{\nu}_w}$ over $\overline{\cup_{i=0}^k B_r(C^i) \cap \Omega}$, and $k \geq 1$, thus $C^k \in \partial\Omega$ by strong maximum principle. From properties $\phi_{\boldsymbol{\nu}_w}$ above: for any minimal chain: $C^k \in \Gamma^0_{shock} \cup \{P_2\}$.

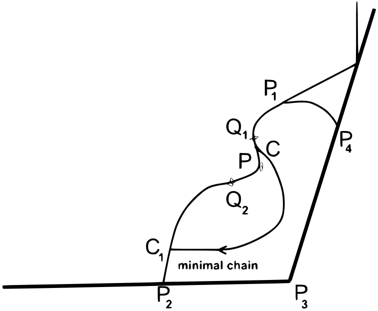


Property 2.

If $A, B \in \Gamma_{shock}$ and $\boldsymbol{\nu}_{sh}(A) = \boldsymbol{\nu}_{sh}(B)$, with $AB \cdot \boldsymbol{\nu}(A) > 0$, then $\phi_{\boldsymbol{\nu}_w}(A) > \phi_{\boldsymbol{\nu}_w}(B)$

Note: on picture, A lies on "convex" part of Γ_{shock}, and B lies on "non-convex" part of Γ_{shock}. This can be used in the argument: minimal chain ends in A, and we further reduce $\phi_{\boldsymbol{\nu}_w}$ if we find such B on a "non-convex" part of Γ_{shock}, then B is not a point of local minimum of $\phi_{\boldsymbol{\nu}_w}$, can start a minimal chain from B. After several steps there is no place for endpoint of chain, a contradiction.

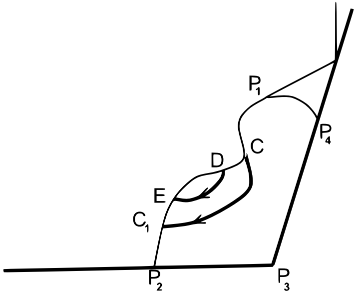
Steps of proof of convexity of Γ_{shock}



Suppose there exists $P \in \Gamma^0_{shock}$ with $\phi_{\tau\tau}(P) < 0$ ("wrong direction of convexity"). Recall $\phi = \varphi - \varphi_1$.

Let Q_1Q_2 be the maximal interval on Γ_{shock} with $\phi_{\tau\tau} < 0$ and $P \in Q_1Q_2$. Then $Q_1Q_2 \subset (\Gamma_{shock})^0$, by monotonicity cone of ϕ. Let $C \in Q_1Q_2$ be such that $\phi_{\boldsymbol{\nu}_w}(C) = \min_{Q_1Q_2} \phi_{\boldsymbol{\nu}_w}$.

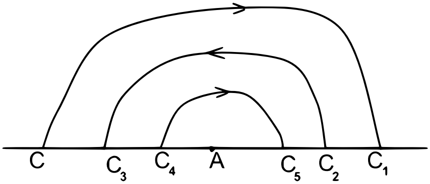
By Property 1, C is not a point of local minimum of $\phi_{\boldsymbol{\nu}_w}$ relative to $\overline{\Omega}$. Then there exists a minimal chain (with r small) starting at C, with endpoint at $C_1 \in \Gamma^0_{shock} \cup \{P_2\}$. Can show $C_1 \neq P_2$.



Then $\phi_{\boldsymbol{\nu}_w}(C_1) < \phi_{\boldsymbol{\nu}_w}(C)$, and $\phi_{\boldsymbol{\nu}_w}$ has a local minimum at C_1. By Property 1, $\phi_{\tau\tau}(C_1) > 0$, i.e. C_1 is on "convex" part of shock.

A contradiction would be obtained, if we show, by Property 2, existence D on CC_1 with $\phi_{\boldsymbol{\nu}_w}(D) = \min_{CC_1} \phi_{\boldsymbol{\nu}_w} < \phi_{\boldsymbol{\nu}_w}(C_1)$ and $\phi_{\tau\tau}(D) \leq 0$. Then there exists a minimal chain from D, it must end at $E \in CC_1$ and $\phi_{\boldsymbol{\nu}_w}(E) < \phi_{\boldsymbol{\nu}_w}(D)$, which contradicts the definition of D.

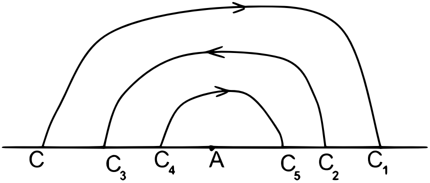
However, to use Property 2, we have to control the directions of $\boldsymbol{\nu}$ on Γ_{shock}. This requires extra steps.



We show $\max_{CC_1} \phi_{\boldsymbol{\nu}_w} > \phi_{\boldsymbol{\nu}_w}(C)$. Then there exists $A \in (CC_1)^0$ such that $\phi_{\boldsymbol{\nu}_w}(A) = \max_{CC_1} \phi_{\boldsymbol{\nu}_w}$. We show: A is a local maximum of $\phi_{\boldsymbol{\nu}_w}$ relative to $\overline{\Omega}$, and $\boldsymbol{\nu}(A) \neq \boldsymbol{\nu}(P)$ for all P in $CC_1 \setminus A$. We can control directions of $\boldsymbol{\nu}$ on subintervals AC and AC_1.

We show, using Property 2, that there exists C_2 on AC_1 with $\phi_{\boldsymbol{\nu}_w}(C_2) = \min_{AC_1} \phi_{\boldsymbol{\nu}_w} < \phi_{\boldsymbol{\nu}_w}(C_1)$ and $\phi_{\tau\tau}(C_2) \leq 0$.

Then there exists a minimal chain from C_2; its endpoint C_3 must be on CC_1 and $\phi_{\boldsymbol{\nu}_w}(C_3) < \phi_{\boldsymbol{\nu}_w}(C_2)$. It follows that $C_3 \in AC$.



Now we show, using Property 2, that there exists C_4 on AC_3 with $\phi_{\boldsymbol{\nu}_w}(C_4) = \min_{AC_3} \phi_{\boldsymbol{\nu}_w} < \phi_{\boldsymbol{\nu}_w}(C_3)$ and $\phi_{\tau\tau}(C_4) \leq 0$.

Then there exists a minimal chain from C_4; its endpoint C_5 must be on C_2C_3 and $\phi_{\boldsymbol{\nu}_w}(C_5) < \phi_{\boldsymbol{\nu}_w}(C_4)$. It follows that $C_5 \in AC_2$. But then

$$\phi_{\boldsymbol{\nu}_w}(C_5) < \phi_{\boldsymbol{\nu}_w}(C_4) < \phi_{\boldsymbol{\nu}_w}(C_3) < \phi_{\boldsymbol{\nu}_w}(C_2) = \min_{AC_1} \phi_{\boldsymbol{\nu}_w},$$

a contradiction.

Open problems

1) Prove existence of regular reflection solutions for Euler system. One of difficulties is in vorticity estimates, noticed by D. Serre for isentropic Euler system: vorticity is not in $L^2(\Omega)$. Singularities are expected near the tip of wedge. Thus one has to work in the low regularity framework: velocity is discontinuous (but subsonic) near tip of wedge. On the positive side, this may improve stability of solutions: For potential flow, regular reflection solution does not exist for non-symmetric perturbations of the incoming flow (J. Hu, 2018) because velocity cannot be subsonic and discontinuous in case of potential flow. For Euler system, existence for non-symmetric perturbations can be expected.

2) Uniqueness/nonuniqueness in various classes of solutions. For example, for reflection of oblique shock by a flat wall, in the class of self-similar solutions for Euler system, etc.

3) Mach reflection: develop apriori estimates.