Divergence-measure fields: Gauss-Green formulas and Normal traces

Monica Torres

Purdue University

Workshop on Stability Analysis of Nonlinear PDEs

August 15-19, 2022

Joint work with Gui-Qiang Chen (University of Oxford), Giovanni Comi (University of Hamburg) and Qinfeng Li (Hunan University).

Research supported by the Simons Foundation and the National Science Foundation

Classical Gauss-Green formula

The Gauss-Green Theorem was motivated by the analysis of fluids, capillarity and potential theory (electrical and gravitational potentials). Implications of the Gauss-Green theorem include the Maxwell's discovery of the laws of electrodynamics. The derivations of the Maxwell's equations or the Euler equations are based on the validity of the Gauss-Green Theorem and the Stokes theorem.

The formula was discovered by Lagrange in 1762, but he did not provide proof of the result. The theorem was rediscovered by Gauss and Ostrogradsky. Ostrogradsky stated and proved the Divergence-Theorem in an article that was presented in 1828 and published in 1831. Ortrogradsky's method of proof was similar to the approach Gauss used in his paper published in 1813 where Gauss used a particular case of the theorem. Independently, Green also rediscovered the Divergence Theorem for $n = 2$, and published his result in 1828. The Divergence Theorem in vector form:

$$\int_U \mathit{div}\, \boldsymbol{F} \,\mathrm{d}x = -\int_{\partial U} \boldsymbol{F} \cdot \nu \,\mathrm{d}\mathcal{H}^{n-1},$$

where $\boldsymbol{F}$ is a C^1 vector field, U is a bounded open set with piecewise smooth boundary, and ν in the **inner** unit normal to U, was later formulated thanks to the development of Vector Calculus.

Gauss-Green formulas for Lipschitz vector fields on sets of finite perimeter

How to extend the Gauss-Green formula to very rough sets? The development of geometric measure theory by De Giorgi and Federer opened the door to the extension of the classical Gauss-Green formula over sets of finite perimeter (whose boundaries can be very rough and contain cusps, corners, etc) and Lipschitz vector fields. Indeed, we can consider the left side of the formula as a lineal functional acting on vector fields $\boldsymbol{F} \in C_c^1(\mathbb{R}^n)$. If E is such that the functional $\boldsymbol{F} \to \int_E \mathit{div}\, \boldsymbol{F}$ is bounded in a particular sense , then the Riesz representation theorem immediately yields a Radon measure, denoted as μ_E, such that

$$\int_E \mathit{div}\, \boldsymbol{F}\, \mathrm{d}x = \int_{\mathbb{R}^n} \boldsymbol{F} \cdot d\mu_E, \text{ for all } \boldsymbol{F} \in C_c^1(\mathbb{R}^n).$$

The Radon measure μ_E is actuallly $-D\chi_E$, where $D\chi_E$ is the distributional gradient of the characteristic function of E, and E is called a set of finite perimeter in $\mathbb{R}^n$.

The structure theorem of De Giorgi

The structure theorem of De Giorgi shows that, even though the boundary of E can be very rough, it also has nice tangential properties which means that there is a notion of measure-theoretic tangent plane. More rigorously said, the topological boundary of E (denoted as ∂E) contains an $(n-1)$-rectifiable set, known as the reduced boundary of E and denoted as $\partial^* E$, which can be covered by a countable union of C^1 surfaces, up to a set of $\mathcal{H}^{n-1}$-measure zero. It can be shown that every $x \in \partial^* E$ has an inner unit normal $\nu_E(x)$ and a tangent plane in the measure-theoretic sense. Moreover, the Radon measure μ_E has the following structure:

$$\mu_E = -\nu_E \mathcal{H}^{n-1} \llcorner \partial^* E,$$

and therefore the previous formula reduces to

$$\int_E \operatorname{div} \boldsymbol{F}\, dx = -\int_{\partial^* E} \boldsymbol{F}(y) \cdot \nu_E(y)\, d\mathcal{H}^{n-1}(y).$$

This Gauss-Green formula for Lipschitz vector fields $\boldsymbol{F}$ over sets of finite perimeter was proved by De Giorgi and Federer in a series of papers that span the period 1945-1958.

Traces and Gauss-Green formulas for Sobolev and BV functions on Lipschitz domains

In various fields of analysis (i.e.; Partial Differential Equations or Calculus of Variations) it is necessary to work with functions which are not Lipschitz, but only in L^p, $1 \leqslant p \leqslant \infty$, whose derivatives in the distributional sense belong to L^p. That is, the corresponding $\boldsymbol{F}$ is a Sobolev vector field. The existence of traces in L^p, defined on the boundary of the domain, and which make the formula valid over open sets with Lipschitz boundary, was studied by Aronszajn, Babich-Slobodetskij and Peodi for $p = 2$. For $p \geqslant 2$, necessary and sufficient conditions for the existence of traces of Sobolev functions were obtained by Gagliardo in 1957, although previous authors had obtained some necessary or sufficient conditions, while other authors were previously aware of the fact that the traces, defined on the boundary of the domain, belonged to L^p.

For many other applications the theory of Sobolev spaces is not sufficient. For example, we note that if E has C^2 boundary then χ_E, belongs to L^1, but the distributional derivate $D\chi_E$ does not belong to L^1, but is in fact a Radon measure and $|D\chi_E|(\Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega)$. Thus, a larger space functions is required, the space of functions of bounded variation (BV), which consists of all functions in L^1 whose distributional derivatives are Radon measures. This space has compactness properties that allow, for instance, to show the existence of minimal surfaces. Moreover, the Gauss-Green formula for BV vector fields over Lipschitz domains holds (Burago-Maz'ya-Vol'pert).

Systems of hyperbolic conservation laws

$$
\begin{aligned}
&\mathbf{u}_t + \operatorname{div}_x \mathbf{f}(\mathbf{u}) = 0, \quad \mathbf{u} \in L^\infty(\mathbb{R}^{d+1}, \mathbb{R}^m) \\
&x \in \mathbb{R}^d, \quad n := d + 1 \\
&\mathbf{f} = (\mathbf{f}^1, \mathbf{f}^2, ..., \mathbf{f}^m), \quad \mathbf{f}^i : \mathbb{R}^m \to \mathbb{R}^d
\end{aligned} \tag{1}
$$

- Solutions develop singularities (shock waves) even if the initial data is smooth.
- If the initial data has small total variation, and $d = 1$, the *Random Choice Method* (Glimm) gives existence of solutions in the space of functions of bounded variation BV. The *Front Tracking Method* (Bressan) is another method to construct BV solutions.
- In general, no existence theory for multidimensional systems.
- Solutions of systems of hyperbolic conservation laws are not, in general, BV functions. If the initial data has large total variation and/or the strict hyperbolicity of the systems fails then solutions are no longer in BV (even for $d = 1$).

Systems of hyperbolic conservation laws

- Rauch (1987) proved that, for dimension greater that 1, in order to have estimates of the form
$$\|\mathbf{u}(t,\cdot)\|_{BV} \leqslant C\,\|\mathbf{u}_0\|_{BV}\,,$$
the flux $\mathbf{f}$ must satisfy the following commutation property
$$\nabla\mathbf{f}_i\nabla\mathbf{f}_j = \nabla\mathbf{f}_j\nabla\mathbf{f}_i \qquad (2)$$
- If $d=1$ or $m=1$ (2) holds. In more dimensions, the inviscid equations of compressible fluid dynamics do not satisfy (2), and therefore it is not possible to obtain BV estimates in these cases. If we consider the nonisentropic case or we add other physical effects the problem becomes even harder.
- However, existence theorems for solutions $\mathbf{u}$ to 2×2 systems (using compensated compacteness) or multidimensional scalar conservation laws show that
  - $\mathbf{u}(t,\mathbf{x}) \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^d)$ or $\mathbf{u}(t,\mathbf{x}) \in L^p(\mathbb{R}_+ \times \mathbb{R}^d)$.
  - $\partial_t\eta(\mathbf{u}) + \mathrm{div}_{\mathbf{x}}\mathbf{q}(\mathbf{u}) \leqslant 0$ in the sense of distributions.

Motivation to study divergence-measure fields

A convex function $\eta : \mathbb{R}^m \to \mathbb{R}$ is an entropy if $\exists\, \mathbf{q} \in C(\mathbb{R}^m, \mathbb{R}^d)$ such that

$$\nabla \mathbf{q}_k(\mathbf{u}) = \nabla \eta(\mathbf{u}) \nabla \mathbf{f}_k(\mathbf{u}), \quad k = 1, 2, ..., d.$$

The pair $(\eta, \mathbf{q})$ is called a convex entropy pair. A bounded entropy solution $\mathbf{u} \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^m)$ of the system (1) is characterized by the entropy inequality

$$\eta(\mathbf{u})_t + \mathrm{div}_x\, \mathbf{q}(\mathbf{u}) \leqslant 0 \text{ in } \mathcal{D}'_{t,x} \tag{3}$$

for any convex entropy pair. If we define

$$\boldsymbol{F}^\eta_{\mathbf{u}}(t,x) := (\eta(\mathbf{u}(t,x)), \mathbf{q}(\mathbf{u}(t,x))),$$

then (3) and the Riesz representation theorem imply that $\exists$ a nonnegative measure $\mu_\eta \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^d)$ such that

$$-\mathit{div}_{t,x}(\eta(\mathbf{u}(t,x)), \mathbf{q}(\mathbf{u}(t,x))) = \mu_\eta; \text{ that is, } \boldsymbol{F}^\eta_{\mathbf{u}} \text{ is a divergence-measure field.} \tag{4}$$

Divergence-measure fields

Definition: Given an integrable vector field $\boldsymbol{F}$ on the open set Ω, *div* $\boldsymbol{F}$ is a distribution acting on $C_c^\infty(\Omega)$ such that, for any test function $\phi \in C_c^\infty(\Omega)$,

$$\langle \mathit{div}\, \boldsymbol{F}, \phi \rangle := -\int_\Omega \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}x. \tag{5}$$

We say that $\boldsymbol{F}$ is an L^p divergence-measure field in the open set Ω for $1 \leqslant p \leq \infty$ if $\boldsymbol{F} \in L^p(\Omega)$ and

$$\sup \left\{ \int_\Omega \boldsymbol{F} \cdot \nabla \phi \, : \, \phi \in C_c^1(\Omega), |\phi| \leq 1 \right\} < \infty. \tag{6}$$

Condition (6) implies that *div* $\boldsymbol{F}$ is finite Radon measure in Ω (*i.e.* $|\mathit{div}\, \boldsymbol{F}|(\Omega) < \infty$) so that

$$\langle \mathit{div}\, \boldsymbol{F}, \phi \rangle = \int_\Omega \phi \, \mathrm{d}\mathit{div}\, \boldsymbol{F} = -\int_\Omega \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}x. \tag{7}$$

The Banach space $\mathcal{DM}^p(\Omega)$ consists of all L^p divergence-measure fields on Ω.

Divergence-measure fields

Sobolev vector fields ($W^{1,1}$)

$\subset$

BV vector fields

$\subset$

Divergence-measure fields

$$\boldsymbol{F} = (f_1, f_2, ..., f_n)$$

$$D\boldsymbol{F} = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 & \dots & \partial_{x_n} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 & \dots & \partial_{x_n} f_2 \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ \partial_{x_1} f_n & \partial_{x_2} f_n & \dots & \partial_{x_n} f_n \end{bmatrix}$$

Example: how to define trace for a bounded divergence-measure field?

Let U be the open unit square in $\mathbb{R}^2$ that has one of its sides contained in the line segment

$$L := \{y = (y_1, y_2) : y_1 = y_2\} \cap \partial U.$$

Define a field $\boldsymbol{F} : \mathbb{R}^2 \setminus L \to \mathbb{R}^2$ by $\boldsymbol{F}(y) = \boldsymbol{F}(y_1, y_2) = \left(\sin\left(\frac{1}{y_1 - y_2}\right), -\sin\left(\frac{1}{y_1 - y_2}\right)\right)$.

div $\boldsymbol{F} = 0$ in the sense of distributions in $\mathbb{R}^n$ so $\boldsymbol{F} \in \mathcal{DM}^\infty(\mathbb{R}^2)$

div $\boldsymbol{F} = 0$ pointwise in $\mathbb{R}^n \setminus L$.

The field is singular on one side, L, of ∂U and therefore, $\boldsymbol{F}$ is undefined on ∂U; it has no trace on ∂U in the classical sense. Note also that the points of L are all essential singularities of $\boldsymbol{F}$ because $\lim_{y \to x} \boldsymbol{F}(y), y \in \mathbb{R}^2 \setminus L, x \in L$ does not exist. Also, $\boldsymbol{F} \notin BV_{loc}(\mathbb{R}^2, \mathbb{R}^2)$

Example

We could try to define the normal trace of $\boldsymbol{F}$ on ∂U as follows:

$$\lim_{t\to 0}\int_{\partial U_t}\boldsymbol{F}(y)\cdot\boldsymbol{\nu}(y)d\mathcal{H}^1(y)=\lim_{t\to 0}\int_{U_t}\textit{div}\,\boldsymbol{F}dy=\lim_{t\to 0}0=0,$$

where $U_t := \{y\in U : \text{dist}(y,\partial U)>t)\}$ or U_t defined in a way that it is smooth and approximates U in a suitable way.

- $\boldsymbol{F}$ has an essential singularity at each point of L and therefore cannot be defined on L;
- We need to make rigorous the above limit and show that $\boldsymbol{F}$ has a weak normal trace on L which is sufficient for the Gauss-Green theorem to hold.

Question: How to approximate a set with smooth sets from the interior and exterior?

Approximation of set of finite perimeter from "inside" and "outside"

The following method works for bounded divergence measure fields:
We consider two representatives:

$$E^1 \text{ and } E^1 \cup \partial^* E$$

$$\chi_k = \chi_E * \rho_{1/k}, \quad A_{k,t} := \{\chi_k > t\}$$

We have

$$|\mathit{div}\,\boldsymbol{F}|(E^1 \Delta A_{k,t}) < \varepsilon \quad t > 1/2, k \geqslant k_*(\varepsilon, t),$$

$$|\mathit{div}\,\boldsymbol{F}|((E^1 \cup \partial^* E)\Delta A_{k,t}) < \varepsilon \quad t < 1/2, k \geqslant k_*(\varepsilon, t),$$

$$\mathcal{H}^{n-1}(\partial A_{k,t} \cap (E^0 \cup \partial^* E)) \to 0, \text{ as } k \to \infty, \text{ for a.e. } t > 1/2.$$

$$\mathcal{H}^{n-1}(\partial A_{k,t} \cap (E^1 \cup \partial^* E)) \to 0, \text{ as } k \to \infty, \text{ for a.e. } t < 1/2.$$

Normal traces for divergence-measure fields

Chen-T-Ziemer:

Theorem: Let $\boldsymbol{F} \in \mathcal{DM}^{\infty}_{loc}(\Omega, \mathbb{R}^n)$ and let $E \Subset \Omega$ be a bounded set of finite perimeter. Then, there exist $\mathcal{F}_i \cdot \boldsymbol{\nu} \in L^{\infty}(\partial^* E)$ such that and $\mathcal{F}_e \cdot \boldsymbol{\nu} \in L^{\infty}(\partial^* E)$ such that

$$\int_{E^1} \varphi \, div \, \boldsymbol{F} + \int_{E^1} \boldsymbol{F} \cdot \nabla \varphi = -\int_{\partial^* E} \varphi \mathcal{F}_i \cdot \boldsymbol{\nu}(\mathbf{y}) d\mathcal{H}^{n-1}(\mathbf{y})$$

and

$$\int_{E^1 \cup \partial^* E} \varphi \, div \, \boldsymbol{F} + \int_{E^1 \cup \partial^* E} \boldsymbol{F} \cdot \nabla \varphi = -\int_{\partial^* E} \varphi (\mathcal{F}_e \cdot \boldsymbol{\nu})(\mathbf{y}) d\mathcal{H}^{n-1}(\mathbf{y}),$$

for every $\varphi \in C^{\infty}_c(\Omega)$.

$$\|\mathfrak{F}_i \cdot \boldsymbol{\nu}\|_{\infty} \leqslant \|\boldsymbol{F}\|_{\infty} \quad \|\mathfrak{F}_e \cdot \boldsymbol{\nu}\|_{\infty} \leqslant \|\boldsymbol{F}\|_{\infty}$$

We want a normal trace $\mathfrak{F} \cdot \boldsymbol{\nu}$ obtained as the limit of the classical normal traces $\boldsymbol{F} \cdot \boldsymbol{\nu}$ defined on almost every smooth surface that approximates $\partial^* E$ from the "inside" or "outside".

Definition of trace as a distribution

Definition: Given $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$, $1 \leq p \leq \infty$, and a bounded Borel set $E \subset \Omega$, we define the normal trace of $\boldsymbol{F}$ on ∂E as

$$\langle \boldsymbol{F} \cdot \nu, \phi \rangle_{\partial E} := \int_E \phi \, d \mathit{div} \boldsymbol{F} + \int_E \boldsymbol{F} \cdot \nabla \phi \, dx, \tag{8}$$

for any $\phi \in \mathrm{Lip}_c(\mathbb{R}^n)$.

Remark: We notice that, by the definition, the normal trace of $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ on the boundary of a bounded Borel set $E \subset \Omega$ is a distribution of order 1 on $\mathbb{R}^n$, since, for any $\phi \in C^1_c(\mathbb{R}^n)$, we have

$$| \langle F \cdot \nu, \phi \rangle_{\partial E} | \leq \|\phi\|_{L^\infty(\mathbb{R}^n)} |\mathit{div} F|(E) + \|\nabla \phi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} |E|^{1-\frac{1}{p}} \|\boldsymbol{F}\|_{L^p(E;\mathbb{R}^n)}.$$

The normal trace is not stable a priori under modifications of E by Lebesgue negligible sets. Indeed, if $\tilde{E}$ is any measurable set such that $|E \Delta \tilde{E}| = 0$, then, unless $|\mathit{div} \boldsymbol{F}| \ll \mathcal{L}^n$, we may have $|\mathit{div} \boldsymbol{F}|(E \Delta \tilde{E}) \neq 0$, even though the second terms are equal.

For bounded divergence-measure fields, the trace is actually a measure, and even better a function

Corollary: Let $\boldsymbol{F} \in \mathcal{DM}^{\infty}_{\mathrm{loc}}(\Omega)$ and $E \Subset \Omega$ be a set of finite perimeter. Then the normal trace of $\boldsymbol{F}$ on the boundary of any Borel (or *div* $\boldsymbol{F}$-measurable) representative $\tilde{E}$ of the set E is a Radon measure supported on $\partial\tilde{E}$.
In addition, the normal traces of $\boldsymbol{F}$ on the boundaries of E^1 and $E^1 \cup \partial^* E$ are Radon measures absolutely continuous with respect to $\mathcal{H}^{n-1} \llcorner \partial^* E$. More precisely, for any set E of locally finite perimeter in Ω and $\phi \in C^0(\Omega)$ such that $\nabla\phi \in L^1_{\mathrm{loc}}(\Omega)$ and $\chi_E \phi$ has compact support in Ω, we have

$$\int_{E^1} \phi \, d\,\mathrm{div}\,\boldsymbol{F} + \int_E \boldsymbol{F} \cdot \nabla\phi \, dx = -\int_{\partial^* E} \phi \, (\mathcal{F}_i \cdot \nu_E) \, d\mathcal{H}^{n-1}, \tag{9}$$

$$\int_{E^1 \cup \partial^* E} \phi \, d\,\mathrm{div}\,\boldsymbol{F} + \int_E \boldsymbol{F} \cdot \nabla\phi \, dx = -\int_{\partial^* E} \phi \, (\mathcal{F}_e \cdot \nu_E) \, d\mathcal{H}^{n-1}, \tag{10}$$

$$\langle \boldsymbol{F} \cdot \nu, \phi \rangle_{\partial E^1} = -\int_{\partial^* E} \phi \, (\mathcal{F}_i \cdot \nu_E) \, d\mathcal{H}^{n-1},$$

$$\langle \boldsymbol{F} \cdot \nu, \phi \rangle_{\partial (E^1 \cup \partial^* E)} = -\int_{\partial^* E} \phi \, (\mathcal{F}_e \cdot \nu_E) \, d\mathcal{H}^{n-1}, \text{ for any } \phi \in \mathrm{Lip}_c(\Omega).$$

A Gauss-Green formula on domains with fractures for bounded divergence-measure fields

Consider for example

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : |x| < 1, x_2 \neq 0\}$$

The above formulas apply but, since

$$\Omega^1 = \{x \in \mathbb{R}^2 : |x| < 1\},$$

the integration does not happen over the disk with a diameter removed. In some applications we may want to integrate on a domain with fractures or cracks, but since the cracks are part of the topological boundary and belong to the measure-theoretic interior Ω^1, we can not use previous formulas. In order to prove a Gauss-Green formula that includes this example, we will work with open sets of finite perimeter Ω satisfying

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^0) < \infty.$$

Therefore, $\partial\Omega$ can still have a large set of cusps or points of density zero (i.e., points belonging to Ω^0).

Example: The trace can be concentrated outside the reduced boundary

Example: Let $\Omega = D \setminus S$, where $D = (-1,1) \times (-1,1)$ and $S = (-1,1) \times \{0\}$. We define

$$\boldsymbol{F}(x_1, x_2) := \begin{cases} (0,1) & \text{for } x_2 > 0, \\ (0,-1) & \text{for } x_2 < 0. \end{cases} \tag{11}$$

Let $\Omega^+ = D \cap \{x_2 > 0\}$ and $\Omega^- = D \cap \{x_2 < 0\}$. We also let $S_1 := (-1,1) \times \{1\}$ and $S_2 := (-1,1) \times \{-1\}$. Then, for any $\phi \in C_c^1(\mathbb{R}^2)$, we have

$$\begin{aligned} \int_\Omega \boldsymbol{F} \cdot \nabla\phi \, \mathrm{d}x &= \int_{\Omega^+} \boldsymbol{F} \cdot \nabla\phi \, \mathrm{d}x + \int_{\Omega^-} \boldsymbol{F} \cdot \nabla\phi \, \mathrm{d}x \\ &= \int_{\Omega^+} \partial_{x_2}\phi \, \mathrm{d}x - \int_{\Omega^-} \partial_{x_2}\phi \, \mathrm{d}x \\ &= \int_S (-\phi) \, \mathrm{d}\mathcal{H}^{n-1} - \int_S \phi \, \mathrm{d}\mathcal{H}^1 + \int_{S_1 \cup S_2} \phi \, \mathrm{d}\mathcal{H}^1 \\ &= -2 \int_S \phi \, \mathrm{d}\mathcal{H}^1 + \int_{S_1 \cup S_2} \phi \, \mathrm{d}\mathcal{H}^1. \end{aligned}$$

where we have used the classical Gauss-Green formula.

Example: The trace can be concentrated outside the reduced boundary

Since $\mathit{div}\,\boldsymbol{F} = 0$ on Ω, the previous computation yields

$$\langle \boldsymbol{F} \cdot \nu,\, \phi \rangle_{\partial\Omega} = \int_\Omega \phi \,\mathrm{d}\mathit{div}\,\boldsymbol{F} + \int_\Omega \boldsymbol{F} \cdot \nabla\phi \,\mathrm{d}x = -2\int_S \phi \,\mathrm{d}\mathcal{H}^1 + \int_{S_1 \cup S_2} \phi \,\mathrm{d}\mathcal{H}^1.$$

Therefore, $\langle \boldsymbol{F} \cdot \nu,\, \cdot \rangle_{\partial\Omega}$ is a measure $\mu := -2\mathcal{H}^1 \llcorner S + \mathcal{H}^1 \llcorner (S_1 \cup S_2)$.
Motivated by this example, in order to study trace $\langle \boldsymbol{F} \cdot \nu,\, \cdot \rangle_{\partial\Omega}$, for a bounded divergence-measure field $\boldsymbol{F}$ and an extension domain Ω, the measure-theoretic interior part of the topological boundary has to be considered. This example has motivated us to study the characterization of domains satisfying

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^0) < \infty,$$

and to formulate and prove the following theorem:

A theorem based on Besicovitch's covering theorem

Definition: $E^\alpha = \{x \in \mathbb{R}^n : \lim_{r \to 0} \frac{E \cap B(x,r)}{|B(x,r)|} = \alpha\}$

Theorem (Chen-Li-T.): Let Ω be a bounded set with $|\Omega| > 0$. Then there exists smooth sets $E_k \Subset \Omega$ such that

$$E_k \to \Omega \text{ in } L^1$$

and

$$\sup_k \mathcal{H}^{n-1}(\partial E_k) < \infty$$

if and only if

$$\mathcal{H}^{n-1}(\partial \Omega \setminus \Omega^0) < \infty.$$

Proof of the approximation theorem (*only if* part)

Let E_i be the assumed approximating sequence. Then, by the lower semicontinuity of $P(\cdot)$ we know that Ω is of finite perimeter. It suffices to show

$$\mathcal{H}^{n-1}(\partial\Omega \cap \Omega^1) < \infty, \tag{12}$$

since $\partial\Omega \setminus \Omega^0 = (\partial\Omega \cap \Omega^1) \cup \partial^m\Omega$. Since $E_i \Subset \Omega$ for each i,

$$\lim_{r\to 0} \frac{|B_r(x) \cap (\Omega \setminus E_i)|}{\omega_n r^n} = 1 \qquad \text{for all } x \in \partial\Omega \cap \Omega^1.$$

Therefore, for any $x \in \partial\Omega \cap \Omega^1$, we can choose $0 < r < \infty$ such that

$$\frac{|B_r(x) \cap (\Omega \setminus E_i)|}{\omega_n r^n} \geqslant \frac{1}{2}.$$

From the relative isoperimetric inequality we have

$$\begin{aligned} P\left(\Omega \setminus E_i; B_r(x)\right) &\geq c(n) \min\left\{|B_r(x) \cap (\Omega \setminus E_i)|^{\frac{n-1}{n}}, |B_r(x) \setminus (\Omega \setminus E_i)|^{\frac{n-1}{n}}\right\} \\ &= c_1(n) r^{n-1}. \end{aligned} \tag{13}$$

Proof of the approximation theorem

Vitali's covering theorem ==> there is a family of countable disjoint balls $B_{r_j}(x_j)$

$$\partial\Omega \cap \Omega^1 \subset \cup_j B_{5r_j}(x_j), \quad \frac{|B_{r_j}(x_j) \cap (\Omega \setminus E_i)|}{\omega_n r_j^n} \geqslant \frac{1}{2}, \quad r_j^{n-1} \lesssim_n P\left(\Omega \setminus E_i; B_{r_j}(x_j)\right) \tag{14}$$

Let $\delta_i = \sup_j r_j$. We have

$$\begin{aligned}
\mathcal{H}^{n-1}_{5\delta_i}(\partial\Omega \cap \Omega^1) &\leq n\omega_n 5^{n-1} \sum_j r_j^{n-1} \\
&\lesssim_n \sum_j P\left(\Omega \setminus E_i; B_{r_j}(x_j)\right) \\
&\leq P(\Omega \setminus E_i) \\
&= P(\Omega) + P(E_i),
\end{aligned} \tag{15}$$

$$\limsup_{i\to\infty} \delta_i \lesssim_n \left(\frac{2}{\omega_n}\right)^{\frac{1}{n}} \limsup_{i\to\infty} |\Omega \setminus E_i|^{\frac{1}{n}} = 0$$

$$\mathcal{H}^{n-1}(\partial\Omega \cap \Omega^1) = \lim_{i\to\infty} \mathcal{H}^{n-1}_{5\delta_i}(\partial\Omega \cap \Omega^1) \lesssim_n P(\Omega) + \limsup_{i\to\infty} P(E_i) < \infty.$$

Proof of the approximation theorem (*If* part)

For any $\delta > 0$ and $x \in \partial\Omega \cap \Omega^0$, we can choose $0 < r < \delta$ such that

$$\frac{|\Omega \cap B_r(x)|}{|B_r(x)|} < \frac{1}{2}. \tag{16}$$

By the relative isoperimetric inequality there is a constant $c(n)$ such that

$$|\Omega \cap B_r(x)|^{\frac{n-1}{n}} \leq c(n) P\left(\Omega; B_r(x)\right). \tag{17}$$

From the coarea formula, we can choose r such that $\mathcal{H}^{n-1}(\partial B_r(x) \cap \partial^m \Omega) = 0$, while (16)–(17) still hold. Therefore, applying the classical Gauss-Green formula to the vector field $\boldsymbol{F}(y) = y - x$ on the set of finite perimeter $\Omega \cap B_r(x)$:

$$\begin{aligned} n|\Omega \cap B_r(x)| &= \int_{\Omega \cap B_r(x)} \mathit{div}_y (y-x) \mathrm{d}y \\ &= - \int_{\Omega^1 \cap \partial B_r(x)} (y-x) \cdot \nu_{B_r(x)}(y) \mathrm{d}\mathcal{H}^{n-1} - \int_{B_r(x) \cap \partial^* \Omega} (y-x) \cdot \nu_\Omega(y) \mathrm{d}\mathcal{H}^{n-1} \\ &\geq r\mathcal{H}^{n-1}(\Omega^1 \cap \partial B_r(x)) - rP\left(\Omega; B_r(x)\right). \end{aligned} \tag{18}$$

Proof of the approximation theorem

$$r\mathcal{H}^{n-1}(\Omega^1 \cap \partial B_r(x)) \leq n|\Omega \cap B_r(x)| + rP\left(\Omega; B_r(x)\right). \tag{19}$$

Moreover, it is clear that

$$\frac{|\Omega \cap B_r(x)|^{\frac{1}{n}}}{r} \leq \omega_n^{\frac{1}{n}}. \tag{20}$$

Combining (17) and (19)–(20), we have

$$\begin{aligned}\mathcal{H}^{n-1}(\Omega^1 \cap \partial B_r(x)) &\leq \frac{n|\Omega \cap B_r(x)|}{r} + P(\Omega; B_r(x)) \\ &= n|\Omega \cap B_r(x)|^{\frac{n-1}{n}} \frac{|\Omega \cap B_r(x)|^{\frac{1}{n}}}{r} + P\left(\Omega; B_r(x)\right) \\ &\leq nc(n)\omega_n^{\frac{1}{n}} P\left(\Omega; B_r(x)\right) + P\left(\Omega; B_r(x)\right),\end{aligned} \tag{21}$$

that is,

$$\mathcal{H}^{n-1}(\Omega^1 \cap \partial B_r(x)) \lesssim_n P\left(\Omega; B_r(x)\right). \tag{22}$$

Proof of the approximation theorem

From Besicovitch's covering theorem, it follows that there exists $\mathcal{F}_i$, $i = 1, 2, \cdots, \xi(n)$, so that each family $\mathcal{F}_i$ contains countably disjoint balls with radius less than δ satisfying

$$\partial\Omega \cap \Omega^0 \subset \cup_{i=1}^{\xi(n)} \cup_{B\in\mathcal{F}_i} B$$

and, for each $B_r(x) \in \cup_{i=1}^{\xi(n)} \mathcal{F}_i$, (22) holds.
Since $\mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^0) < \infty$ there exists a family $\mathcal{F}_0$ of balls such that

$$\sup_{B\in\mathcal{F}_0} \text{diam}(B) \le 2\delta, \tag{23}$$

$$\partial\Omega \setminus \Omega^0 \subset \cup_{B\in\mathcal{F}_0} B, \tag{24}$$

$$\sum_{B\in\mathcal{F}_0} \mathcal{H}^{n-1}(\partial B) \lesssim_n \mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^0). \tag{25}$$

We may also require that, for any $B_r(x) \in \mathcal{F}_0$, $\mathcal{H}^{n-1}(\partial B_r(x) \cap \partial^*\Omega) = 0$.

Since there are countably many balls in $\cup_{i=0}^{\xi(n)} \mathcal{F}_i$, we can assume that

$\mathcal{H}^{n-1}(\partial B_r(x) \cap \partial^*\Omega) = 0$ holds for any $B_r(x) \in \cup_{i=0}^{\xi(n)} \mathcal{F}_i$.

Proof of the approximation theorem

Since $\partial\Omega$ is compact a finite collection of balls $\{B_{r_k}(z_k)\}_{k=1}^N \subset \cup_{i=0}^{\xi(n)} \mathcal{F}_i$ cover $\partial\Omega$.
Let $E = \Omega \setminus \cup_{k=1}^N B_{r_k}(z_k)$ so that $E \Subset \Omega$.

$$\begin{aligned}
P(E) &= P\left(\Omega \setminus \cup_{k=1}^N B_{r_k}(z_k)\right) \\
&= P\left(\cup_{k=1}^N B_{r_k}(z_k); \Omega^1\right) + P\left(\Omega; \mathbb{R}^n \setminus \cup_{k=1}^N B_{r_k}(z_k)\right) \\
&= P\left(\cup_{k=1}^N B_{r_k}(z_k); \Omega^1\right) \\
&\leq \sum_{k=1}^N P(B_{r_k}(z_k); \Omega^1) \\
&\leq \sum_{i=1}^{\xi(n)} \sum_{B \in \mathcal{F}_i} \mathcal{H}^{n-1}(\partial B \cap \Omega^1) + \sum_{B \in \mathcal{F}_0} \mathcal{H}^{n-1}(\partial B \cap \Omega^1),
\end{aligned}$$

$$\begin{aligned}
P(E) \lesssim_n \sum_{i=1}^{\xi(n)} \sum_{B \in \mathcal{F}_i} P(\Omega; B) + \mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^0) \quad &\lesssim_n \quad \xi(n) P(\Omega) + \mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^0) \\
&\lesssim_n \quad \mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^0).
\end{aligned}$$

Proof of the approximation theorem

Since $0 < r < \delta$ for any $B_r(x)$ in the cover of $\partial\Omega$, we can estimate

$$
\begin{aligned}
|\Omega \setminus E| &\leq \sum_{i=0}^{\xi(n)} \sum_{B \in \mathcal{F}_i} |B \cap \Omega| \\
&\lesssim_n \sum_{i=1}^{\xi(n)} \sum_{B \in \mathcal{F}_i} |B \cap \Omega|^{\frac{1}{n}} |B \cap \Omega|^{\frac{n-1}{n}} + \delta \sum_{B \in \mathcal{F}_0} \mathcal{H}^{n-1}(\partial B) \\
&\lesssim_n \delta \Bigg(\sum_{i=1}^{\xi(n)} \sum_{B \in \mathcal{F}_i} P(\Omega; B) + \mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^0) \Bigg) \\
&\lesssim_n \delta \ \left(\xi(n) P(\Omega) + \mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^0) \right) \\
&\lesssim_n \delta \, \mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^0),
\end{aligned}
$$

where we have used the fact that the balls in $\mathcal{F}_i$ are disjoint for $1 \leq i \leq \xi(n)$. Since $|\Omega| > 0$, the previous construction shows that, for each $\delta > 0$ small, we can construct a set $E_\delta \neq \emptyset$ such that

$$E_\delta \Subset \Omega, \qquad |\Omega \setminus E_\delta| \lesssim_n \delta \, \mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^0), \qquad P(E_\delta) \lesssim_n \mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^0)$$

Extension domains for bounded divergence-measure fields

Fiven $\boldsymbol{F} \in \mathcal{DM}^\infty(\Omega)$, the extension of $\boldsymbol{F}$ is defined as

$$\tilde{\boldsymbol{F}}(x) := \begin{cases} \boldsymbol{F}(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \notin \Omega. \end{cases} \tag{26}$$

Definition: We say that Ω is an extension domain for bounded divergence-measure fields if, for any $\boldsymbol{F} \in \mathcal{DM}^\infty(\Omega)$, $\tilde{\boldsymbol{F}}$ is a divergence-measure field in $\mathbb{R}^n$; and

$$|\mathit{div}\,\tilde{\boldsymbol{F}}|(\mathbb{R}^n) < \infty. \tag{27}$$

Theorem: If Ω is a bounded open set satisfying

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^0) < \infty,$$

then Ω is an extension domain for bounded divergence-measure fields.

A Gauss-Green formula up-to the boundary on bounded open sets that can contain fractures

Theorem (Chen-Li-T.:) Let Ω be a bounded extension domain for $\boldsymbol{F} \in \mathcal{DM}^\infty(\Omega)$, and let $\tilde{\boldsymbol{F}}$ be the extension of $\boldsymbol{F}$. Then the trace operator $\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial\Omega}$ is a finite Radon measure μ concentrated on $\partial\Omega \setminus \Omega^0$ with

$$\mu = -\mathit{div}\,\tilde{\boldsymbol{F}} \llcorner \left((\partial\Omega \cap \Omega^1) \cup \partial^*\Omega\right) = -\mathit{div}\,\tilde{\boldsymbol{F}} \llcorner (\partial\Omega \cap \Omega^1) - 2\overline{\tilde{\boldsymbol{F}} \cdot D\chi_\Omega}, \tag{28}$$

where $\overline{\tilde{\boldsymbol{F}} \cdot D\chi_\Omega}$ is a measure concentrated on $\partial^*\Omega$, which is the weak star limit of the sequence of measures $\tilde{\boldsymbol{F}} \cdot \nabla(\chi_\Omega * \rho_\epsilon)$. As a consequence,

$$\mathit{div}\,\tilde{\boldsymbol{F}} \llcorner \partial^*\Omega = 2\overline{\tilde{\boldsymbol{F}} \cdot D\chi_\Omega}. \tag{29}$$

Moreover, there exists $g \in L^1\left(\partial\Omega \setminus \Omega^0; \mathcal{H}^{n-1}\right)$ such that

$$\int_{\mathbb{R}^n} \phi \, \mathrm{d}\mu = \int_{\partial\Omega \setminus \Omega^0} g\phi \, \mathrm{d}\mathcal{H}^{n-1}. \tag{30}$$

A Gauss-Green formula up-to the boundary on bounded open sets that can contain fractures

In particular, if Ω is a bounded open set satisfying

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^0) < \infty,$$

then the above results also hold. Moreover,

$$g \in L^\infty\left(\partial\Omega \setminus \Omega^0; \mathcal{H}^{n-1}\right), \tag{31}$$

and the following Gauss-Green formula up to the boundary holds:

$$\int_\Omega \phi \, \mathrm{d}\mathit{div}\, \boldsymbol{F} + \int_\Omega \boldsymbol{F} \cdot \nabla\phi \, \mathrm{d}x = \int_{\partial\Omega\setminus\Omega^0} g(x) \, \mathrm{d}\mathcal{H}^{n-1}(x)$$

An example of an unbounded vector field

Let

$$\boldsymbol{F}(x) = \frac{x}{|x|^n}, \quad \boldsymbol{F} \in \mathcal{DM}^p_{loc}(\mathbb{R}^n), \quad 1 \leqslant p < \frac{n}{n-1}, \quad \mathit{div}\, \boldsymbol{F} = n\omega_n \delta_0$$

If $n = 2$ we have that $\mathit{div}\, \boldsymbol{F} = 2\pi\delta_0$. Let $U = (0,1) \times (0,1)$.

$$\boldsymbol{F}(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right) \in \mathcal{DM}^p_{loc}(\mathbb{R}^2), 1 \leqslant p < 2$$

$$\int_0^1 \int_0^1 \mathit{div}\, \boldsymbol{F} dx dy = 0 \neq \int_{\partial\Omega} \boldsymbol{F} \cdot \boldsymbol{\nu} d\mathcal{H}^1 = -\frac{\pi}{2},$$

We approximate U with domains U^ε, U_ε from the interior and exterior respectively. For example, we can use the standard signed distance function

$$U^\varepsilon = \{(x,y) \in U : \text{dist}((x,y), \partial U) > \varepsilon\} \quad U_\varepsilon = \{(x,y) \in U : \text{dist}((x,y), \partial U > -\varepsilon\}$$

$$\int_U \mathit{div}\, \boldsymbol{F} = 0 = -\lim_{\varepsilon \to 0} \int_{\partial U^\varepsilon} \boldsymbol{F} \cdot \boldsymbol{\nu} d\mathcal{H}^1, \quad \int_{\overline{U}} \mathit{div}\, \boldsymbol{F} = 2\pi = -\lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon} \boldsymbol{F} \cdot \boldsymbol{\nu} d\mathcal{H}^1$$

A product rule for divergence-measure fields in $\mathcal{DM}^p$, $1 \leqslant p \leqslant \infty$

Theorem: If $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$, $1 \le p < \infty$ and $\phi \in C^0(\Omega) \cap L^\infty(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega; \mathbb{R}^n)$, $p' = \frac{p}{p-1}$, then we have $\phi\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$. In addition, we have

$$\mathit{div}\,(\phi \boldsymbol{F}) = \phi\,\mathit{div}\,\boldsymbol{F} + \boldsymbol{F} \cdot \nabla\phi.$$

Representation of the interior normal trace

Theorem: Chen-Comi-T.[interior normal trace].
Let $U \subset \Omega$ be a bounded open set and $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$, $1 \leqslant p \leqslant \infty$. Then, for any $\phi \in C^0(\Omega) \cap L^\infty(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega;\mathbb{R}^n)$, there exists a set $\mathcal{N}$ with $\mathcal{L}^1(\mathcal{N}) = 0$ such that, for every sequence $\{\varepsilon_k\}$ satisfying $\varepsilon_k \to 0$ and $\varepsilon_k \notin \mathcal{N}$, we have the following representation for the interior normal trace on ∂U

$$< \boldsymbol{F}\cdot\nu, \phi >_{\partial U} = \int_U \phi \, d\mathit{div}\,\boldsymbol{F} + \int_U \boldsymbol{F}\cdot\nabla\phi \, dx = -\lim_{k\to+\infty}\int_{\partial^* U^{\varepsilon_k}} \phi \boldsymbol{F}\cdot\nu_{U^{\varepsilon_k}} \, d\mathcal{H}^{n-1},$$

where $\nu_{U^{\varepsilon_k}}$ is the inner unit normal to U^{ε_k} on $\partial^* U^{\varepsilon_k}$.
In addition, (33) holds also for any open set $U \subset \Omega$, provided that $\mathrm{supp}(\phi)$ is compact in Ω.

Remark: In particular, the previous Theorem implies that, if Ω is bounded, then one can take $U = \Omega$ in (33), thus obtaining a Gauss-Green formula up to the boundary of the open set where $\boldsymbol{F}$ is defined.

Representation of the exterior normal trace

Theorem [exterior normal trace].

Let $U \Subset \Omega$ be an open set and $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$, $1 \leqslant p \leqslant \infty$. Then, for any $\phi \in C^0(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega; \mathbb{R}^n)$, there exists a set $\mathcal{N}$ with $\mathcal{L}^1(\mathcal{N}) = 0$ such that, for every sequence $\{\varepsilon_k\}$ satisfying $\varepsilon_k \to 0$ and $\varepsilon_k \notin \mathcal{N}$, we have the following representation for the exterior normal trace on ∂U

$$< \boldsymbol{F} \cdot \nu, \phi >_{\partial \overline{U}} = \int_{\overline{U}} \phi \, d\mathrm{div}\, \boldsymbol{F} + \int_{\overline{U}} \boldsymbol{F} \cdot \nabla\phi \, dx = - \lim_{k \to +\infty} \int_{\partial^* U_{\varepsilon_k}} \phi \boldsymbol{F} \cdot \nu_{U_{\varepsilon_k}} \, d\mathcal{H}^{n-1},$$

where $\nu_{U_{\varepsilon_k}}$ is the inner unit normal to U_{ε_k} on $\partial^* U_{\varepsilon_k}$.

In addition, (34) holds also for any open set U satisfying $\bar{U} \subset \Omega$, provided that $\mathrm{supp}(\phi)$ is compact in Ω.

Approximations with smooth sets

The previous two main theorems can also be improved to the case where the approximating sets U^ε and U_ε are smooth.

Approximations with smooth sets in the case where U is a C^0 domain

For domains with this regularity we can use the regularized distance ρ instead of the standard signed distance. This distance is C^∞ and was introduced by Lieberman (Pac. J. Math., 1985) for Lipschitz domains and adapted to C^0 domains by Ball-Zarnescu (Calc. Var. & PDE).

$$U^{\varepsilon,\rho} := \{x \in \mathbb{R}^n : \rho(x) > \varepsilon\}$$

and

$$U_{\varepsilon,\rho} := \{x \in \mathbb{R}^n : \rho(x) > -\varepsilon\}.$$

First Green's identity

Theorem : Let $u \in W^{1,p}(\Omega)$ for $1 \le p \le \infty$ be such that $\Delta u \in \mathcal{M}(\Omega)$, and let $U \subset \Omega$ be a bounded open set. Then, for any $\phi \in C^0(\Omega) \cap L^\infty(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega; ^n)$, there exists a set $\mathcal{N} \subset$ with $\mathcal{L}^1(\mathcal{N}) = 0$ such that, for every nonnegative sequence $\{\varepsilon_k\} \not\subset \mathcal{N}$ satisfying $\varepsilon_k \to 0$ as $k \to \infty$,

$$\int_U \phi \, \mathrm{d}\Delta u + \int_U \nabla u \cdot \nabla \phi \, \mathrm{d}x = -\lim_{k\to\infty} \int_{\partial^* U^{\varepsilon_k}} \phi \nabla u \cdot \nu_{U^{\varepsilon_k}} \, \mathrm{d}\mathcal{H}^{n-1}, \tag{32}$$

where $\nu_{U^{\varepsilon_k}}$ is the inner unit normal to U^{ε_k} on $\partial^* U^{\varepsilon_k}$.
In particular, if $u \in W^{1,2}(\Omega) \cap C^0(\Omega) \cap L^\infty(\Omega)$ with $\Delta u \in \mathcal{M}(\Omega)$,

$$\int_U u \, \mathrm{d}\Delta u + \int_U |\nabla u|^2 \, \mathrm{d}x = -\lim_{k\to\infty} \int_{\partial^* U^{\varepsilon_k}} u \nabla u \cdot \nu_{U^{\varepsilon_k}} \, \mathrm{d}\mathcal{H}^{n-1}. \tag{33}$$

In addition, (32) holds also for any open set $U \subset \Omega$, provided that $\mathrm{supp}(\phi) \cap U^\delta \Subset \Omega$ for any small $\delta > 0$. Analogously, (33) holds for any open set $U \subset \Omega$, provided that $\mathrm{supp}(u) \cap U^\delta \Subset \Omega$ for any $\delta > 0$.

Second Green's identity

Corollary: Let $u \in W^{1,p}(\Omega) \cap C^0(\Omega) \cap L^\infty(\Omega)$ and $v \in W^{1,p'}(\Omega) \cap C^0(\Omega) \cap L^\infty(\Omega)$ for $1 \leq p \leq \infty$ be such that $\Delta u, \Delta v \in \mathcal{M}(\Omega)$, and let $U \subset \Omega$ be a bounded open set. Then there exists a set $\mathcal{N} \subset$ with $\mathcal{L}^1(\mathcal{N}) = 0$ such that, for every nonnegative sequence $\{\varepsilon_k\} \not\subset \mathcal{N}$ satisfying $\varepsilon_k \to 0$ as $k \to \infty$,

$$\int_U v \, \mathrm{d}\Delta u - u \, \mathrm{d}\Delta v = - \lim_{k\to\infty} \int_{\partial^* U^{\varepsilon_k}} (v \nabla u - u \nabla v) \cdot \nu_{U^{\varepsilon_k}} \, \mathrm{d}\mathcal{H}^{n-1}, \tag{34}$$

where $\nu_{U^{\varepsilon_k}}$ is the inner unit normal to U^{ε_k} on $\partial^* U^{\varepsilon_k}$. In addition, (34) holds also for any open set $U \subset \Omega$, provided that $\mathrm{supp}(u), \mathrm{supp}(v) \cap U^\delta \Subset \Omega$ for any small $\delta > 0$.

Necessary and sufficient conditions for the trace to be a measure

Theorem: Let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$, $1 \leq p \leq \infty$. Then the normal trace of $\boldsymbol{F}$ on the boundary of a Borel set $E \Subset \Omega$ is a Radon measure supported on ∂E if and only if *div*$(\chi_E \boldsymbol{F}) \in \mathcal{M}(\Omega)$.

Proof. First we show that the distribution $< \boldsymbol{F} \cdot \nu, \cdot >_E$ is supported on ∂E. Let $V \Subset \Omega \setminus \partial E$ and $\phi \in C_c^\infty(V)$. We need to show that $< \boldsymbol{F} \cdot \nu, \phi >_E = 0$.
We have that $\phi \boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ and $\mathrm{supp}(\phi \boldsymbol{F}) \subset V$, which implies $\mathrm{supp}(\textit{div}(\phi \boldsymbol{F})) \subset V$. From this, it follows that

$$< \boldsymbol{F} \cdot \nu, \phi >_E = \textit{div}(\phi \boldsymbol{F})(E) = \textit{div}(\phi \boldsymbol{F})(V \cap E^\circ).$$

We can assume that $E^\circ \neq \emptyset$, otherwise there is nothing to prove, and also that $V \subset E^\circ$, without loss of generality. Then, *div*$(\phi \boldsymbol{F})(V) = 0$, and so the distribution is supported on ∂E.
As for the equivalence, we notice that

$$< \boldsymbol{F} \cdot \nu, \phi >_E - \int_E \phi \, d\textit{div} \boldsymbol{F} = \int_E \boldsymbol{F} \cdot \nabla \phi \, dx = \int_\Omega \chi_E \boldsymbol{F} \cdot \nabla \phi \, dx,$$

for any $\phi \in \mathrm{Lip}_c(\Omega)$. Hence, since *div*$\boldsymbol{F} \in \mathcal{M}(\Omega)$, it follows that $< \boldsymbol{F} \cdot \nu, \cdot >_E \in \mathcal{M}(\partial E)$ if and only if *div*$(\chi_E \boldsymbol{F}) \in \mathcal{M}(\Omega)$, by the density of $\mathrm{Lip}_c(\Omega)$ in $C_c(\Omega)$ with respect to the sup norm. $\square$

An example of trace measure

$\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ admits a normal trace on the boundary of a Borel set $E \Subset \Omega$ representable by a Radon measure if and only if $\mathit{div}(\chi_E \boldsymbol{F}) \in \mathcal{DM}^p(\Omega)$. This condition is generally weaker than the requirement of E to be a set of locally finite perimeter in Ω. Indeed, there exist a set $E \subset \mathbb{R}^2$ with $\chi_E \notin BV_{\mathrm{loc}}(\mathbb{R}^2)$ and a field $\boldsymbol{F} \in \mathcal{DM}^p(\mathbb{R}^2)$ for any $p \in [1, \infty]$ with $\langle \boldsymbol{F} \cdot \nu, \cdot \rangle_{\partial E} \in \mathcal{M}(\partial E)$. The key observation in the construction of such a set E is that, given a constant vector field $\boldsymbol{F} \equiv \mathbf{v} \in \mathbb{R}^n$, the normal trace is given by

$$\langle \mathbf{v} \cdot \nu, \cdot \rangle_{\partial E} = -\mathit{div}(\chi_E \mathbf{v}) = -\sum_{j=1}^{n} v_j D_{x_j} \chi_E.$$

Clearly, the requirement that $\sum_{j=1}^{n} v_j D_{x_j} \chi_E \in \mathcal{M}(\Omega)$ is weaker than the requirement that $\chi_E \in BV(\Omega)$, since there may be some cancellations.
We choose E as the open bounded set whose boundary is given by

$$\partial E = \big(\{0\} \times [0,1]\big) \cup \big([0,1] \times \{0\}\big) \cup \big([0, 1 + \log 2] \times \{1\}\big) \cup S,$$

where

Example: Continuation

$$S = \Big(\{1\} \times [0, \frac{1}{2}]\Big) \bigcup \Big([1,2] \times \{\frac{1}{2}\}\Big) \bigcup \Big(\bigcup_{n \geq 1} \{1 + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\} \times [1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}]\Big)$$

$$\bigcup \Big(\bigcup_{n \geq 1} [1 + \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k}, 1 + \sum_{k=1}^{2n+1} \frac{(-1)^{k-1}}{k}] \times \{1 - \frac{1}{2^{2n+1}}\}\Big)$$

$$\bigcup \Big(\bigcup_{n \geq 1} [1 + \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k}, 1 + \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k}] \times \{1 - \frac{1}{2^{2n}}\}\Big).$$

Then $\chi_E \notin BV_{\text{loc}}(\mathbb{R}^2)$, since $\mathcal{H}^1(S) = \infty$. However, we can show that $D_{x_1}\chi_E \in \mathcal{M}(\mathbb{R}^2)$.

If $\boldsymbol{F}(x_1, x_2) = f(x_2)g(x_1)(1,0)$ for some $f \in L^p(\mathbb{R})$ and $g \in C^1_c(\mathbb{R})$, then $\boldsymbol{F} \in \mathcal{DM}^p(\mathbb{R}^2)$,

$$\textit{div}\, \boldsymbol{F} = f(x_2)g'(x_1)\mathcal{L}^2,$$

and

$$\textit{div}(\chi_E \boldsymbol{F}) = f(x_2)g(x_1)D_{x_1}\chi_E + \chi_E(x_1, x_2)f(x_2)g'(x_1)\mathcal{L}^2. \tag{35}$$

An example (by M. Šilhavý) where trace is not a measure

$$\boldsymbol{F}(x,y) := \frac{(-y,x)}{x^2+y^2}.$$

div $\boldsymbol{F} = 0$, $\boldsymbol{F} \in \mathcal{DM}^p_{\text{loc}}(\mathbb{R}^2)$ for any $1 \le p < 2$, $E = (-1,1) \times (-1,0)$,
$\phi \in \text{Lip}_c((-1,1)^2)$. Then we have

$$\begin{aligned}
\int_{(-1,1)^2} \chi_E \boldsymbol{F} \cdot \nabla\phi \, dx\, dy &= \int_{-1}^{1}\int_{-1}^{0} \frac{1}{x^2+y^2}\left(-y\frac{\partial\phi}{\partial x} + x\frac{\partial\phi}{\partial y}\right) dy\, dx \\
&= \lim_{\varepsilon\to 0}\left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1}\right)\int_{-1}^{0} \frac{1}{x^2+y^2}\left(-y\frac{\partial\phi}{\partial x} + x\frac{\partial\phi}{\partial y}\right) dy\, dx \\
&= \lim_{\varepsilon\to 0}\int_{-1}^{0} \frac{y}{\varepsilon^2+y^2}(-\phi(-\varepsilon,y)+\phi(\varepsilon,y))\, dy + \\
&- \int_{-1}^{0}\left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1}\right)\phi(x,y)\frac{2xy}{(x^2+y^2)^2}\, dx\, dy + \\
&\left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1}\right)\phi(x,0)\frac{1}{x}\, dx + \int_{-1}^{0}\left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1}\right)\phi(x,y)\frac{2xy}{(x^2+y^2)^2}\, dx\, dy \\
&= \text{p.v.}\int_{-1}^{1}\phi(x,0)\frac{1}{x}\, dx,
\end{aligned}$$

An example where trace is not a measure

Thus $\mathit{div}(\chi_E \boldsymbol{F}) \notin \mathcal{M}((-1,1)^2)$, which means $\chi_E \boldsymbol{F} \notin \mathcal{DM}^p((-1,1)^2)$ for any $1 \le p < 2$. The argument can be easily generalized to

$$\boldsymbol{F}(x,y) = \frac{(-y,x)}{(x^2+y^2)^{\frac{\alpha}{2}}},$$

for $2 \le \alpha < 3$, obtaining

$$\mathit{div}(\chi_E \boldsymbol{F}) = (\mathrm{p.v.}\, \mathrm{sgn}(x)\, |x|^{1-\alpha}) \llcorner (-1,1) \otimes \delta_0.$$

A class of vector fields whose trace is a measure

Definition: Given a closed set S in $\mathbb{R}^n$, the $(n-1)$-dimensional Minkowski content is defined as

$$\mathcal{M}_*^{n-1}(S) := \liminf_{\varepsilon \to 0} \frac{|S + B(0,\varepsilon)|}{2\varepsilon}.$$

Proposition: Let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$, $1 \leq p \leq \infty$, and let $U \subset \Omega$ be a bounded open set such that $\mathcal{M}_*^{n-1}(\partial U) < \infty$. Let us assume also that *div* $\boldsymbol{F}$ has compact support in U and that

$$\boldsymbol{F}(x) = \frac{1}{n\omega_n} \int_\Omega \frac{(x-y)}{|x-y|^n} \, d\mathit{div}\, \boldsymbol{F}(y) \qquad (36)$$

for $\mathcal{L}^n$-a.e. $x \in \Omega$. Then, $< \boldsymbol{F} \cdot \nu, \cdot >_{\partial U} \in \mathcal{M}(\partial U)$.

Definition of Lipschitz deformable boundary (Chen-Frid):

Let Ω be an open subset in $\mathbb{R}^n$. We say that $\partial\Omega$ is a deformable Lipschitz boundary provided that the following hold.

(i): For each $x \in \partial\Omega$, there exists $r > 0$ and a Lipschitz mapping $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, upon rotating and relabeling the coordinate axis if necessary,

$$\Omega \cap Q(x,r) = \{y \in \mathbb{R}^n : y_n > \gamma(y_1, ..., y_{n-1})\} \cap Q(x,r),$$

where $Q(x,r) = \{y \in \mathbb{R}^n : |y_i - x_i| \leqslant r, i = 1, ..., n\}$. We denote by $\tilde{\gamma}$ the map $y' = (y_1, ..., y_{n-1}) \to (y', \gamma(y'))$.

(ii): There exists a map $\Psi : \partial\Omega \times [0,1] \to \overline{\Omega}$ such that Ψ is a bi-Lipschitz homeomorphism over its image and $\Psi(\cdot, 0) \equiv \mathrm{Id}$, where Id is the identity map over $\partial\Omega$.
Denote $\partial\Omega_\tau = \Psi(\partial\Omega \times \{\tau\}), \tau \in [0,1]$, and denote Ω_τ the open subset of Ω whose boundary is $\partial\Omega_\tau$. We call Ψ a Lipschitz deformation of $\partial\Omega$.

The Lipschitz deformation is regular if

$$\lim_{\tau \to 0^+} J^{\partial\Omega}\Psi_\tau = 1 \text{ in } L^1(\partial\Omega; \mathcal{H}^{n-1}), \text{ where } \Psi_\tau(x) = \Psi(x,\tau). \quad (37)$$

Theorem (Chen-Comi-T.): Any Lipschitz domain has Lipschitz deformable boundary in the sense of Definition Chen-Frid.

Proof: We follow the construction in Ball-Zarnescu, Calculus of Variations and Partial Differential Equations, 56(1), jan 2017

Theorem (Chen-Comi-T.): If U is a open bounded set with Lipschitz boundary in $\mathbb{R}^n$, then there exists a REGULAR Lipchitz deformation in the sense of Definition Chen-Frid.

Proof: We follow the construction in Nečas (Czechoslovak Mathematical Journal, 1962 and1964) and Verchota (Layer potentials and boundary value problems for the Laplace equation in Lipschitz domains, Ph D thesis, 1982).

Formulas on Lipschitz domains

An immediate consequence of the existence of such Lipschitz diffeomorphism between ∂U and $\partial U^{\varepsilon,\rho}$ or $\partial U_{\varepsilon,\rho}$ is that the area formula can be employed in order to consider only integrals on ∂U.

Theorem: Let $U \Subset \Omega$ be an open set with Lipschitz boundary, let $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ for $1 \le p \le \infty$, and let $\phi \in C^0(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega;^n)$. Then there exists a set $\mathcal{N} \subset$ with $\mathcal{L}^1(\mathcal{N}) = 0$ such that, for every nonnegative sequence $\{\varepsilon_k\} \not\subset \mathcal{N}$ satisfying $\varepsilon_k \to 0$,

$$\int_U \phi \,\mathrm{d}\mathit{div}\,\boldsymbol{F} + \int_U \boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x = -\lim_{k\to\infty}\int_{\partial U}\left(\phi\boldsymbol{F}\cdot\frac{\nabla\rho}{|\nabla\rho|}\right)(\Psi_\tau(x))J^{\partial U}\Psi_\tau(x)\,\mathrm{d}\mathcal{H}^{n-1}, \quad (38)$$

and

$$\int_{\overline{U}} \phi \,\mathrm{d}\mathit{div}\,\boldsymbol{F} + \int_U \boldsymbol{F}\cdot\nabla\phi\,\mathrm{d}x = -\lim_{k\to\infty}\int_{\partial U}\left(\phi\boldsymbol{F}\cdot\frac{\nabla\rho}{|\nabla\rho|}\right)(\Psi_\tau(x))J^{\partial U}\Psi_\tau(x)\,\mathrm{d}\mathcal{H}^{n-1}, \quad (39)$$

In addition, (38) holds also for any bounded open set U with Lipschitz boundary if $\phi \in L^\infty(\Omega)$, and even for an unbounded open set U with Lipschitz boundary if $\mathrm{supp}(\phi) \cap U^\delta \Subset \Omega$ for any $\delta > 0$. Similarly, (39) also holds for any open set U satisfying $\overline{U} \subset \Omega$, provided that $\mathrm{supp}(\phi)$ is compact in Ω.

Some relevant bibliography

G. ANZELLOTTI: Pairings between measures and functions and compensated compactness, Anal. Mat. Pure Appl. **135** (1983), 293–318.

L. AMBROSIO, G. CRIPPA AND S. MANIGLIA: Traces and fine properties of a BD class of vector fields and applications, Ann. Fac. Sci. Toulouse Math. **XIV** (2005), 527–561.

G.-Q. CHEN AND H. FRID: Divergence-measure fields and hyperbolic conservation laws, Arch. Ration. Mach. Anal. **147** (1999), 89–118.

G.-Q. CHEN AND H. FRID: Extended divergence-measure fields and the Euler equations for gas dynamics, Comm. Math. Phys. **236** (2003), 251–280.

G.-Q. CHEN AND M. TORRES: Divergence-measure fields, sets of finite perimeter, and conservation laws, Arch. Ration. Mech. Anal. **175** (2005), 245–267.

Some relevant bibliography

G.-Q. Chen, M. Torres: On the structure of solutions of nonlinear hyperbolic systems of conservation laws, Commun. Pure Appl. Anal. **10**, (2011), 1011–1036.

G.-Q. Chen, M. Torres and W.P. Ziemer: Gauss-Green Theorem for Weakly Differentiable Vector Fields, Sets of Finite Perimeters, and Balance Laws, Comm. Pure Appl. Math. **62** (2009), 242–304.

G. E. Comi and K. R. Payne: On locally essentially bounded divergence measure fields and sets of locally finite perimeter, Adv. in Calc. of Variations, 2018.

G. Q. Chen, G. E. Comi and M. Torres: Cauchy fluxes and Gauss-Green formulas for divergence-measure fields over general open sets, ARMA, 2019.

G. Q. Chen, Q. Li and M. Torres: Traces and extensions of bounded divergence-measure fields in rough domains, Submitted, 2018.

G. E. Comi and M. Torres: One sided approximations of sets of finite perimeter, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 28(1):181-190, 2017.

M. Degiovanni, A. Marzocchi and A. Musesti: Cauchy fluxes associated with tensor fields having divergence measure, Arch. Ration. Mech. Anal. **147** (1999), 197–223.

Some relevant bibliography

H. FRID: Divergence-measure fields on domains with Lipschitz boundary in "Hyperbolic conservation laws and related analysis with applications", 207-225, Springer Proc. Math. Stat. **49**, Springer, Heidelberg, 2014.

B. KAWOHL AND F. SCHURICHT: Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem, Commun. Contemp. Math. **9** (2007), no. 4, 515-543

N.C. PHUC AND M. TORRES: Characterizations of the existence and removalbe singularities of divergence-measure vector fields, Indiana Univ. Math. J. **57** (2008), 1573–1598.

C. SCHEVEN AND T. SCHMIDT: BV supersolutions to equations of 1-Laplace and minimal surface type, J. Differential Equations **261** (2016), no. 3, 1904-1932.

F. SCHURICHT: A new mathematical foundation for contact interactions in continuum physics, Arch. Ration. Mech. Anal. **184** (2007), 495–551.

M. ŠILHAVÝ: Divergence measure fields and Cauchy's stress theorem, Rend. Sem. Mat. Univ. Padova **113** (2005), 15–45.

T. DE PAUW AND W. PFEFFER: DIVERGENCE THEOREM FOR UNBOUNDED VECTOR FIELDS. TRANS. AMER. MATH. SOC. 359 (2007), NO. 12, 5915-5929

T. DE PAUW: ON THE DIVERGENCE THEOREM, AUTOUR DU CENTENAIRE LEBESGUE, 85-121, PANOR. SYNTHÈSES, 18, SOC. MATH. FRANCE, PARIS, 2004.

T. DE PAUW AND W. PFEFFER: THE GAUSS-GREEN THEOREM AND REMOVABLE SETS FOR PDES IN DIVERGENCE FORM. ADV. MATH. 183 (2004), NO. 1, 155-182.