Self-similar solutions of the 1d-Landau-Lifshitz-Gilbert equation and related problems

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The Landau–Lifshitz–Gilbert equation

The 1d Landau–Lifshitz–Gilbert equation (LLG)

$$\partial_t \mathbf{m} = \underbrace{\beta \mathbf{m} \times \mathbf{m}_{ss}}_{\text{exchange interaction}} \underbrace{- \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{ss})}_{\text{dissipative term}}, \quad s \in \mathbb{R}, \quad t \in I \subseteq \mathbb{R}, \qquad \text{(LLG)}$$

- $\mathbf{m} = (m_1, m_2, m_3) : \mathbb{R} \times I \longrightarrow \mathbb{S}^2$ is the magnetization vector
- α is the Gilbert damping coefficient
- $\alpha, \ \beta \in [0, 1]$ such that $\alpha^2 + \beta^2 = 1$

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The family of LLG- equations includes the well-known geometric evolution equations

- $\alpha = 0$: Schrödinger map
$$\partial_t \mathbf{m} = \mathbf{m} \times \mathbf{m}_{ss}$$
- $\alpha = 1$: Heat flow for harmonic maps into $\mathbb{S}^2$
$$\partial_t \mathbf{m} = \mathbf{m}_{ss} + \mathbf{m}|\mathbf{m}_s|^2$$
- $0 < \alpha < 1$: LLG interpolates between both models. LLG is a hybrid between these two models.

The Landau-Lifshitz-Gilbert equation

$$\partial_t \mathbf{m} = \beta \mathbf{m} \times \mathbf{m}_{ss} - \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{ss}), \quad s \in \mathbb{R}, \quad t \in I \subseteq \mathbb{R}, \qquad \text{(LLG)}$$

- Properties:
  - $\frac{\partial}{\partial t} |\mathbf{m}|^2 = 0, \forall t.$
  - Scale invariance: $\forall \lambda > 0, \quad \mathbf{m}_\lambda(s, t) = \mathbf{m}(\lambda s, \lambda^2 t).$
  - Rotation invariance: $\mathbf{m}_{\mathcal{R}}(s, t) = \mathcal{R}\mathbf{m}(s, t)$ for all $\mathcal{R} \in SO(3)$.
  - LLG and 1d Cubic dissipative Schrödinger equations (via the Hasimoto transformation and stereographic projection).
  - Time-reversibility:
    - $\alpha = 0$ SM is time-reversible
    - $\alpha \in (0, 1]$ is not time-reversible. LLG is of parabolic type.

Self-similar solutions of the 1d LLG equation

$$\mathbf{m}(s,t) = \mathbf{m}(\lambda s, \lambda^2 t), \qquad \forall \lambda > 0, \qquad s \in \mathbb{R}, \quad t \in \mathbb{R}^+ \quad \text{or} \quad t \in \mathbb{R}^-.$$

- Expander: $\mathbf{m}(s,t) = \mathbf{m}\left(\frac{s}{\sqrt{t}}\right), \quad (s,t) \in \mathbb{R} \times (0,\infty)$
- Shrinker: $\mathbf{m}(s,t) = \mathbf{m}\left(\frac{s}{\sqrt{-t}}\right), \quad (s,t) \in \mathbb{R} \times (-\infty,0) \quad$ for $\quad \mathbf{m} : \mathbb{R} \longrightarrow \mathbb{S}^2.$

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Motivation:

- Expanders evolve from a singular value at time $T = 0$. Expanders are related to non-uniqueness phenomena, resolution of singularities and long time description of solutions.
- Shrinkers evolve towards a singular value at time $T = 0$. Shrinkers are often related to phenomena of singularity formation.
- The understanding of the dynamics and properties of self-similar solutions also provide an idea of which are the **natural spaces** to develop a well-posedness theory that captures these often physically relevant structures.

Aim:

Existence and analytical study of **self-similar solutions of the $1d$-LLG equation** with emphasis on the behaviour of these solutions with respect to the damping parameter $\alpha \in [0, 1]$.

In the 1-dimensional case:

- For the Schödinger map ($\alpha = 0$): Lakshmanan, Buttke, G-Rivas-Vega, Vega-Banica.
- Little is known analytically about the effect of damping on the evolution of a one-dimensional spin chain.

Rigidity result

If $\mathbf{m}$ regular solution of LLG

$$\partial_t \mathbf{m} = \beta \mathbf{m} \times \mathbf{m}_{ss} - \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{ss}), \quad s \in \mathbb{R}, \tag{LLG}$$

of the form

$$\mathbf{m}(s,t) = \mathbf{m}\left(\frac{s}{\sqrt{-t}}\right) \qquad \text{or} \qquad \mathbf{m}(s,t) = \mathbf{m}\left(\frac{s}{\sqrt{t}}\right)$$

for some $\mathbf{m} : \mathbb{R} \longrightarrow \mathbb{S}^2$, then $\mathbf{m}$ solves

$$\pm \frac{s}{2} \mathbf{m}' = \beta \mathbf{m} \times \mathbf{m}'' - \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{m}'')$$

which recasts as

$$\pm \frac{s}{2} \mathbf{m}' = \beta \mathbf{m} \times \mathbf{m}'' + \alpha |\mathbf{m}'|^2 \mathbf{m} + \alpha \mathbf{m}'' \tag{$\star$}$$

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Rigidity result: If $\mathbf{m}$ is a regular solution of $(\star)$, then

$$|\mathbf{m}'(s)| = c_0 e^{\pm \alpha s^2/4}, \qquad \text{for some} \quad c_0 \geq 0.$$

$$\partial_t \mathbf{m} = \beta \mathbf{m} \times \mathbf{m}_{ss} - \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{ss}), \quad s \in \mathbb{R}, \quad t \geq 0, \qquad \text{(LLG)}$$

- Geometric representation of the LLG equation

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- Geometric representation of the LLG equation

Let us suppose that $\mathbf{m}$ is the tangent vector of a curve in $\mathbb{R}^3$, that is

$$\mathbf{m} = \mathbf{X}_s \qquad \text{for some} \qquad \mathbf{X}(s,t) \in \mathbb{R}^3$$

parametrized by arc-length and with curvature c and torsion τ.

By the Serret–Frenet formulae, the curve satisfies

$$\begin{cases} \mathbf{m}_s = c\mathbf{n}, \\ \mathbf{n}_s = -c\mathbf{m} + \tau\mathbf{b}, \\ \mathbf{b}_s = -\tau\mathbf{n}, \end{cases} \qquad \text{(SF)}$$

c: curvature, τ : torsion, $\mathbf{n}, \mathbf{b}$: unitary normal, binormal vectors.

$$\partial_t \mathbf{m} = \beta \mathbf{m} \times \mathbf{m}_{ss} - \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{ss}), \quad s \in \mathbb{R}, \quad t \geq 0, \qquad \text{(LLG)}$$

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From (SF), we have

$$\mathbf{m}_{ss} = c_s \mathbf{n} + c(-\mathbf{m} + \tau \mathbf{b}),$$

and thus (LLG) rewrites as

$$\partial_t \mathbf{m} = \beta(c_s \mathbf{b} - c\tau \mathbf{n}) + \alpha(c\tau \mathbf{b} + c_s \mathbf{n}) \quad (geo - LLG)$$

Expanders. G-de Laire (2015)

- We are interested in self-similar solutions of (LLG) of the form

$$(\star) \qquad \mathbf{m}(s,t) = \mathbf{m}\left(\frac{s}{\sqrt{t}}\right) \quad \text{for some profile} \quad \mathbf{m} : \mathbb{R} \longrightarrow \mathbb{S}^2.$$

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- Then, $\mathbf{n}$, $\mathbf{b}$, c and τ are also self-similar.
- If $\mathbf{m}(s,t)$ of the form $(\star)$ solves

$$\partial_t \mathbf{m} = \beta(c_s \mathbf{b} - c\tau \mathbf{n}) + \alpha(c\tau \mathbf{b} + c_s \mathbf{n}), \quad (geo - LLG)$$

then, using (SF)

$$-\frac{s}{2} c\mathbf{n} = \beta(c'\mathbf{b} - c\tau\mathbf{n}) + \alpha(c\tau\mathbf{b} + c'\mathbf{n}),$$

$$\Downarrow$$

$$-\frac{s}{2} c = \alpha c' - \beta c\tau \quad \text{and} \quad \beta c' + \alpha c\tau = 0.$$

$$\Downarrow$$

$$c(s) = c_0 \, e^{-\frac{\alpha s^2}{4}} \quad \text{and} \quad \tau(s) = \frac{\beta s}{2}.$$

Existence

Let $\alpha \in [0,1]$ and $c_0 > 0$.

There exists a unique solution $\{\mathbf{m}_{c_0,\alpha}, \mathbf{n}_{c_0,\alpha}, \mathbf{b}_{c_0,\alpha}\} \in (\mathcal{C}^\infty(\mathbb{R};\mathbb{S}^2))^3$ solution of the Serret-Frenet equations with

$$c(s) = c_0 e^{-\alpha \frac{s^2}{4}} \qquad \text{and} \qquad \tau(s) = \beta \frac{s}{2}$$

and

$$\mathbf{m}(0) = (1,0,0), \quad \mathbf{n}(0) = (0,1,0), \quad \mathbf{b}(0) = (0,0,1), \quad (IC)$$

- Define $$\mathbf{m}_{c_0,\alpha}(s,t) = \mathbf{m}_{c_0,\alpha}\left(\frac{s}{\sqrt{t}}\right).$$

$\mathbf{m}_{c_0,\alpha}(\cdot,t)$ is a regular $\mathcal{C}^\infty(\mathbb{R};\mathbb{S}^2)$ solution of the LLG equation for all $t > 0$.

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- How does $\mathbf{m}_{c_0,\alpha}(s,t)$ behave for t small?
- How does the presence of damping affect the dynamical behaviour of these solutions for positive times close to zero?

$$\mathbf{m}_{c_0,\alpha}(s,t) = \mathbf{m}_{c_0,\alpha}\left(\frac{s}{\sqrt{t}}\right).$$

Understanding $\quad \mathbf{m}_{c_0,\alpha}(s,t) \quad$ as $\quad t \to 0^+ \Longleftrightarrow$

Understanding the associated profile $\quad \mathbf{m}_{c_0,\alpha}(s) \quad$ as $\quad s \to +\infty \Longleftrightarrow$

Integrate the S-F equations with $c(s) = c_0 e^{-\alpha s^2/4}, \tau(s) = \beta \frac{s}{2}$.

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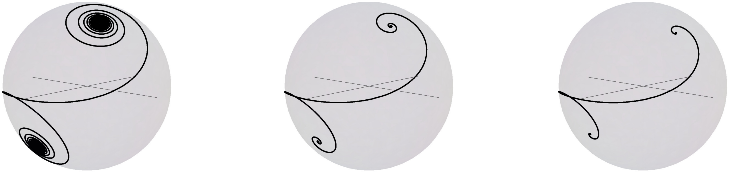

Figure: The profile $\mathbf{m}_{c_0,\alpha}$ for $c_0 = 0.8$ and different values of $\alpha = 0.01, 0.2, 0.4$.

Integration of the S-F system: Reduction to a second order ODE

Let $\mathbf{m} = (m_j(s))$, $\mathbf{n} = (n_j(s))$, $\mathbf{b} = (b_j(s))$ be a solution of the (SF) equations with

$$c(s) = c_0 e^{-\alpha \frac{s^2}{4}} \quad \text{and} \quad \tau(s) = \beta \frac{s}{2}, \quad c_0 > 0.$$

Change of variables related to the stereographic projection. Reduction to a Ricatti equation (Struik 1961). Nonlinear change of function.

$$f''(s) + \frac{s}{2}(\alpha + i\beta) f'(s) + \frac{c_0^2}{4} e^{-\alpha s^2/2} f(s) = 0$$

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- $\{m_j, n_j, b_j\}$ in terms of f_j:

$$m_1(s) = 2|f_1(s)|^2 - 1, \qquad n_1(s) + i b_1(s) = \frac{4}{c_0} e^{\alpha s^2/4} \overline{f}_1(s) f_1'(s),$$

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The second order complex equation. Asymptotics

Fixed $c_0 > 0$, $\alpha \in [0,1]$ and β s.t. $\alpha^2 + \beta^2 = 1$. Consider

$$f''(s) + \frac{s}{2}(\alpha + i\beta) f'(s) + \frac{c_0^2}{4} e^{-\alpha s^2/2} f(s) = 0,$$

$\alpha = 0$: Explicit solution (parabolic cylinder functions). Fourier Analysis tech.

$\alpha = 1$: Explicit solution (involving trigonometric functions of the error function).

$\alpha \in (0,1)$: **Approach:**

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$\alpha \in (0, 1)$: **Approach:**

- Define the new real-valued variables:

$$z = |f|^2, \qquad y = \mathrm{Re}(\bar{f} f'), \qquad h = \mathrm{Im}(\bar{f} f')$$

- Study the system of equations satisfied by the new variables.

- Technique: Integral asymptotics. Use the oscillatory character of the solutions to obtain bounds <u>independent on $\alpha \in [0, 1]$</u>.

Theorem

Let $\alpha \in [0,1]$, $c_0 > 0$, *and* $\mathbf{m}_{c_0,\alpha}$ *be as before. Then, there exist* $\mathbf{A}^{+}_{c_0,\alpha}, \mathbf{B}^{+}_{c_0,\alpha} \in \mathbb{S}^2$ *such that for all* $s \geq s_0 = 4\sqrt{8 + c_0^2}$*:*

$$\mathbf{m}_{c_0,\alpha}(s) = \mathbf{A}^{+}_{c_0,\alpha} - \frac{2c_0}{s}\mathbf{B}^{+}_{c_0,\alpha} e^{-\alpha s^2/4}(\alpha \sin(\vec{\phi}(s)) + \beta \cos(\vec{\phi}(s))) + l.o.t$$

$$\phi_j(s) = a_j + \beta \int_{s_0^2/4}^{s^2/4} \sqrt{1 + c_0^2 \frac{e^{-2\alpha\sigma}}{\sigma}}\, d\sigma, \quad a_j \in [0, 2\pi), \quad j \in \{1,2,3\}$$

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- $\alpha = 0$: $$\phi_j(s) = a_j + \frac{s^2}{4} + c_0^2 \ln(s) + C(c_0) + O\left(\frac{1}{s^2}\right)$$

  There is a logarithmic contribution in the oscillation.

- $\alpha \in (0,1)$: $$\phi_j(s) = a_j + \frac{\beta s^2}{4} + C(\alpha, c_0) + O\left(\frac{e^{-\alpha s^2/2}}{\alpha s^2}\right)$$

- $\alpha = 1$: $$\phi_j(s) = a_j$$

Theorem

$$\mathbf{m}_{c_0,\alpha}(s,t) = \mathbf{m}_{c_0,\alpha}\left(\frac{s}{\sqrt{t}}\right), \qquad t > 0.$$

(i) *The function* $\mathbf{m}_{c_0,\alpha}(s,t)$ *is a regular* $\mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^+;\mathbb{S}^2)$*-solution of (LLG) for* $t > 0$.

(ii) *(Convergence as* $t \to 0^+$*) There exist unitary vectors* $\mathbf{A}^{\pm}_{c_0,\alpha}$ *such that*

$$\lim_{t\to 0^+} \mathbf{m}_{c_0,\alpha}(s,t) = \begin{cases} \mathbf{A}^{+}_{c_0,\alpha}, & \text{if } s > 0, \\ \mathbf{A}^{-}_{c_0,\alpha}, & \text{if } s < 0, \end{cases} \tag{1}$$

with $\mathbf{A}^{-}_{c_0,\alpha} = (A^{+}_{1,c_0,\alpha}, -A^{+}_{2,c_0,\alpha}, -A^{+}_{3,c_0,\alpha})$.

(iii) *(Rate of convergence) For* $t > 0$ *and* $p \in (1,\infty)$

$$\|\mathbf{m}_{c_0,\alpha}(\cdot,t) - \mathbf{A}^{+}_{c_0,\alpha}\chi_{[0,\infty)}(\cdot) - \mathbf{A}^{-}_{c_0,\alpha}\chi_{(-\infty,0)}(\cdot)\|_{L^p(\mathbb{R})} \le C t^{\frac{1}{2p}}, \tag{2}$$

(iv) *Precise asymptotic behaviour for times close to* 0 *given by the asymptotic behaviour of the profile.*

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$\alpha = 0$: $A^{+}_{1,c_0,0} = e^{-\frac{\pi c_0^2}{2}}$ [G-Rivas-Vega] $\implies \mathbf{A}^{+}_{c_0,0} \neq \mathbf{A}^{-}_{c_0,0}$ $\quad c_0 > 0$

$\alpha = 1$: $A^{+}_{1,c_0,1} = \cos(c_0\sqrt{\pi})$ $\implies \mathbf{A}^{+}_{c_0,1} \neq \mathbf{A}^{-}_{c_0,1}$ $\quad c_0 \neq k\sqrt{\pi}, \quad k \in \mathbb{N}$.

$\alpha \in (0,1)$: No explicit formulae for $\mathbf{A}^{+}_{c_0,\alpha}$.

- The map $(c_0, \alpha) \to \mathbf{A}^{+}_{c_0,\alpha}$ is continuous.
- Behaviour of $\mathbf{A}^{+}_{c_0,\alpha}$ for a fixed $\alpha \in (0,1)$ and "small" $c_0 > 0$:

$$A^{+}_{2,c_0,\alpha}, A^{+}_{3,c_0,\alpha} \sim c_0 \frac{\sqrt{\pi(1 \pm \alpha)}}{\sqrt{2}} \implies \mathbf{A}^{+}_{c_0,\alpha} \neq \mathbf{A}^{-}_{c_0,\alpha} \quad \text{small } c_0 > 0$$

The map "$c_0 \longrightarrow \mathbf{A}^{\pm}_{c_0,\alpha}$" for fixed $\alpha \in [0,1]$

For fixed $\alpha \in [0,1]$, consider the map:

$$\boxed{c_0 \longrightarrow \theta_{c_0,\alpha}} \qquad \theta_{c_0,\alpha} = \text{angle}(\mathbf{A}^{+}_{c_0,\alpha}, -\mathbf{A}^{-}_{c_0,\alpha})$$

- Surjectivity: Does $\theta_{c_0,\alpha}$ attain any value in $[0,\pi]$ by varying the parameter c_0?
- Injectivity: Can we generate the same angle using different values of c_0?

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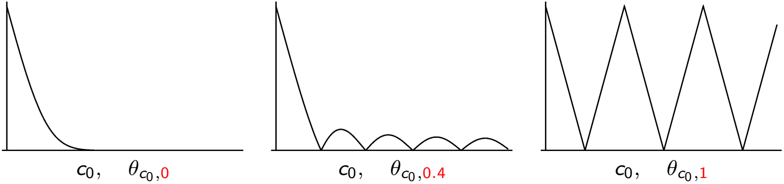

- Any angle is attained.

- Non-uniqueness phenomena for fixed $\alpha \in (0,1]$.

- $[\mathbf{A}^{+}_{c_0,\alpha} = \mathbf{A}^{-}_{c_0,\alpha} \Leftrightarrow \theta_{c_0,\alpha} = \pi] \Longrightarrow$ fixed $\alpha \in [0,1)$, $\quad \mathbf{A}^{+}_{c_0,\alpha} \neq \mathbf{A}^{-}_{c_0,\alpha}$, $\quad \forall c_0 > 0$.

The analytical and numerical analysis carried out on self-similar expanders suggests that:

Given unitary vectors $\mathbf{A}^+$, $\mathbf{A}^-$, one should be able to show the existence of solution for LLG with initial condition:

$$\mathbf{m}^0(s) = \mathbf{A}^+ \chi_{\mathbb{R}^+}(s) + \mathbf{A}^- \chi_{\mathbb{R}^-}(s) \qquad \text{step function} \qquad ,$$

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Step functions are in BMO

$$\mathbf{f}(x) = \mathbf{A}\chi_{\mathbb{R}_-}(x) + \mathbf{B}\chi_{\mathbb{R}_+}(x) \implies [\mathbf{f}]_{BMO} = |\mathbf{B} - \mathbf{A}| \quad (\textit{The jump!})$$

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Approach: Adapt and extend the techniques developed by Koch-Lamm and Wang for $\alpha = 1$ to prove a global well-posedness result for LLG with $\alpha \in (0, 1]$ for data $\mathbf{m}^0$ in $L^\infty(\mathbb{R}^N; \mathbb{S}^2)$ with small BMO semi-norm.

Theorem (G-de Laire, 2019)

Let $\alpha \in (0,1]$ *Given* $\mathbf{m}^0 = (m_1^0, m_2^0, m_3^0) \in L^\infty(\mathbb{R}^N; \mathbb{S}^2)$ *satisfying*

$$\mathbf{m}^0 \quad \textit{away from South Pole} \qquad \textit{and} \quad [\mathbf{m}^0]_{BMO} \leq \varepsilon, \qquad (\textit{sufficiently small}),$$

then there exists a unique solution $\mathbf{m} = (m_1, m_2, m_3) \in X$ *of (LLG) such that*

$$\mathbf{m} \quad \textit{away from South Pole} \qquad \textit{and} \quad [\mathbf{m}]_X \leq K_2 \qquad (\textit{living in "small" ball of } X).$$

Moreover, i) $\mathbf{m} \in \mathcal{C}^\infty(\mathbb{R}^N \times \mathbb{R}^+; \mathbb{S}^2)$, *ii)* $\mathbf{m}(\cdot, t) \to \mathbf{m}^0$ *in* $S'(\mathbb{R}^N)$ *as* $t \to 0^+$,

$$iii) \qquad \|\mathbf{m} - \mathbf{n}\|_X \leq C\, \|\mathbf{m}^0 - \mathbf{n}^0\|_{L^\infty}$$

$$X = \{\, \mathbf{m} : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{S}^2 : \|\mathbf{m}\|_X := \sup_{t>0} \|\mathbf{m}(t)\|_{L^\infty} + [\mathbf{m}]_X < \infty \},$$

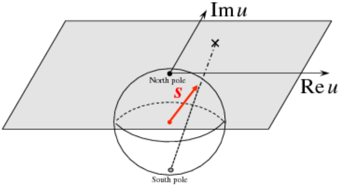
$$[\mathbf{m}]_X := \sup_{t>0} \sqrt{t} \|\nabla \mathbf{m}\|_{L^\infty} + \sup_{\substack{x \in \mathbb{R}^N \\ r>0}} \left(\frac{1}{r^N} \int_{B_r(x)\times[0,r^2]} |\nabla \mathbf{m}(y,t)|^2 \, dy\, dt \right)^{\frac{1}{2}},$$

Idea of the proof

- Stereographic projection: relation between LLG and a dissipative quasilinear Schrödinger equation.

  Let **m** be a solution of (LLG). Using the stereographic variable

$$u = \mathcal{P}(\mathbf{m}) = \frac{m_1 + im_2}{1 + m_3},$$



$$i\partial_t u + (\beta - i\alpha)\Delta u = 2(\beta - i\alpha)\frac{\bar{u}}{1+|u|^2}(\nabla u)^2 := g(u), \qquad \text{(DNLS)}$$

- Fixed point technique: Duhamel formulation

$$u(t) = S_\alpha(t)u^0 - i\int_0^t S_\alpha(t-s)g(u(s))\,ds,$$

  where $S_\alpha(t) = e^{(\alpha+i\beta)t\Delta}$.

- Transfer the estimates back: Good estimates for the mapping $\mathcal{P}$ and $\mathcal{P}^{-1}$ in the space X

$$i\partial_t u + (\beta - i\alpha)\Delta u = 2(\beta - i\alpha)\frac{\bar{u}}{1+|u|^2}(\nabla u)^2 := g(u), \qquad \text{(DNLS)}$$

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Remark: We see that $|g(u)| \leq |\nabla u|^2$, so we need to control $|\nabla u|^2$.

- Koch and Tataru considered the spaces BMO and BMO^{-1} well-adapted to $S_1(t)$.

- Our contribution: introduce the spaces BMO_α and BMO_α^{-1} adapted to $S_\alpha(t)$

$$[f]_{BMO_\alpha} := \sup_{\substack{x\in\mathbb{R}^N \\ r>0}} \left(\frac{1}{r^N}\int_{Q_r(x)} |\nabla S_\alpha(t)f|^2\,dt\,dy \right)^{\frac{1}{2}},$$

$$\|f\|_{BMO_\alpha^{-1}} := \sup_{\substack{x\in\mathbb{R}^N \\ r>0}} \left(\frac{1}{r^N}\int_{Q_r(x)} |S_\alpha(t)f|^2\,dt\,dy \right)^{\frac{1}{2}}, \qquad Q_r(x) = B_r(x)\times[0,r^2].$$

Some Consequences

1. **The Cauchy problem for the 1d-LLG equation with a jump initial data:**

$$\begin{cases} \partial_t \mathbf{m} = \beta \mathbf{m} \times \mathbf{m}_{ss} - \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{ss}), & \text{on} \quad \mathbb{R} \times \mathbb{R}^+, \\ \mathbf{m}^0_{\mathbf{A}^\pm} := \mathbf{A}^+ \chi_{\mathbb{R}^+} + \mathbf{A}^- \chi_{\mathbb{R}^-}, & \end{cases} \qquad \text{(1d LLG-jump)}$$

where $\mathbf{A}^\pm \in \mathbb{S}^2$ with the **angle** between $\mathbf{A}^+$ and $\mathbf{A}^-$ is **small** ($[\mathbf{m}^0_{\mathbf{A}^\pm}]_{BMO}$ small).

(a) The solution of (1d LLG-jump) given by our theorem is a rotation of a self-similar solution $\mathbf{m}_{c_0,\alpha}$ for an appropriate value of c_0.

(b) For any given $\mathbf{m}^0 \in \mathbb{S}^2$ satisfying the hypothesis of our theorem and close enough to $\mathbf{m}^0_{\mathbf{A}^\pm}$ in the L^∞-norm, the corresponding solution must remain "close" to a rotation of a self-similar solution $\mathbf{m}_{c_0,\alpha}$, for some $c_0 > 0$.

2. **Existence of self-similar solutions in higher dimensions:** If in addition $\mathbf{m}^0$ is homogeneous of degree zero, then the solution is a self-similar solution.

3. **Improvement of previous known results:** $\|\nabla \mathbf{m}^0\|_{BMO^{-1}} \simeq \|\mathbf{m}^0\|_{BMO}$

$$\underbrace{L^N(\mathbb{R}^N)}_{Melcher} \subseteq \underbrace{M^{2,2}(\mathbb{R}^N)}_{Lin-Lai-Wang} \subseteq BMO^{-1}(\mathbb{R}^N)$$

4. **Other initial data:**

$$\mathbf{m}^0(x) = (e^{ia \log|x|}(A_1 + iA_2), A_3), \qquad a \quad \textbf{small}.$$

Shrinkers. G-de Laire (2020)

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- We are interested in self-similar solutions of (LLG) of the form

$$\mathbf{m}(s,t) = \mathbf{m}\left(\frac{s}{\sqrt{-t}}\right). \quad \text{for some profile} \quad \mathbf{m} : \mathbb{R} \longrightarrow \mathbb{S}^2.$$

Identifying the profile with the tangent vector of a curve parametrized w.r.t arclengh...

$$c(s) = c\, e^{\alpha \frac{s^2}{4}} \quad \text{and} \quad \tau(s) = -\beta \frac{s}{2}.$$

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$$c(s) = c\, e^{\alpha \frac{s^2}{4}} \quad \text{and} \quad \tau(s) = -\beta \frac{s}{2}.$$

Let $\alpha \in (0,1)$ and $c > 0$. There exists a unique solution of the Serret-Frenet equations $\{\mathbf{m}_{c,\alpha}, \mathbf{n}_{c,\alpha}, \mathbf{b}_{c,\alpha}\} \in (\mathcal{C}^\infty(\mathbb{R}; \mathbb{S}^2))^3$ with c and τ as above and

$$\mathbf{m}_{c,\alpha}(0) = (1,0,0), \quad \mathbf{n}_{c,\alpha}(0) = (0,1,0), \quad \mathbf{m}_{c,\alpha}(0) = (0,0,1). \quad (IC)$$

- Define $$\mathbf{m}_{c,\alpha}(s,t) = \mathbf{m}_{c,\alpha}\left(\frac{s}{\sqrt{-t}}\right) \qquad t < 0.$$

$\mathbf{m}_{c,\alpha}(\cdot, t)$ is a regular $\mathcal{C}^\infty(\mathbb{R}; \mathbb{S}^2)$ solution of the LLG equation for all $t < 0$.

- How does the solution behaves at t approaches the singularity time $T = 0$?

- Analysis of the asymptotic behaviour of the profile: Direct analysis of the Serret-Frenet.

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Figure: Profile $\mathbf{m}_{c,\alpha}$ for $c = 0.5$ and $\alpha = 0.5$.

- The **profile** oscillates in a plane passing through the origin whose normal vector is given by $\mathbf{B}^{\pm} = \lim_{s\to\pm\infty} \mathbf{b}_{c,\alpha}(s)$ as $s \to \pm\infty$ resp.

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- The trajectories of **solutions** form limit circles on the sphere contained in the planes passing through the origin with normal vectors given by the limit at infinity of the associated binormal vector.
- Shrinkers provide examples of blow-up in finite time, where the singularity develops due to rapid oscillations of the profile.

$$\lim_{t\to 0^-} |\partial_s \mathbf{m}_{c,\alpha}(s,t)| = \lim_{t\to 0^-} \frac{c}{\sqrt{-t}} e^{\frac{\alpha s^2}{4(-t)}} = \infty, \qquad \forall s \in \mathbb{R}$$

Some Perspectives

- Shape preserving solutions for 1d-LLG

$$\mathbf{m}(s,t) = e^{\frac{\mathcal{A}}{2}\log t}\mathbf{m}\left(\frac{s}{\sqrt{t}}\right), \qquad \mathcal{A} = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad a \neq 0$$

- Explore how to use this geometric approach in higher dimensions. For example, to study radial solutions of the N-dimensional LLG, Heat Flow for Harmonic Maps and Schrödinger maps.

- Well-posedness theory in the case $\alpha = 0$.

- Other dissipative non-linear Schrödinger equations including gradients.

Thank you for your attention!