The one-sided ergodic Hilbert transform

Markus Haase*

Oxford, September 4, 2009

*joint work with Yuri Tomilov

Power-bounded operators

standing assumptions:

X Banach space, $T \in \mathcal{L}(X)$, $M := \sup_{n \geq 0} \|T^n\| < \infty$.

and: $\overline{\operatorname{ran}}(\mathrm{I} - T) = X$

Power-bounded operators

standing assumptions:

X Banach space, $T \in \mathcal{L}(X)$, $M := \sup_{n \geq 0} \|T^n\| < \infty$.

and: $\overline{\mathrm{ran}}(\mathrm{I} - T) = X$

Then:

$$A_n x := \frac{1}{n+1} \sum_{j=0}^{n} T^j x \;\to\; 0 \qquad (x \in X).$$

Power-bounded operators

standing assumptions:

X Banach space, $T \in \mathcal{L}(X)$, $M := \sup_{n \geq 0} \| T^n \| < \infty$.

and: $\overline{\mathrm{ran}}(\mathrm{I} - T) = X$

Then:

$$A_n x := \frac{1}{n+1} \sum_{j=0}^{n} T^j x \;\to\; 0 \qquad (x \in X).$$

Rate of convergence?

- no rate in general
- $A_n x = o(1/n) \quad \to \quad x = 0$
- $A_n x = O(1/n) \quad \to \quad x \in \mathrm{ran}(\mathrm{I} - T)$

[Butzer-Westphal 1971]

Starting Point

[Derriennic-Lin 2001]

$0 < s < 1, \quad (1-z)^s = \sum_{n=0}^{\infty} a_n^{(s)} z^n, \quad \to \quad a^{(s)} \in \ell^1$

$(\mathrm{I} - T)^s := \sum_{n=0}^{\infty} a_n^{(s)} T^n$

$$x \in \operatorname{ran}(\mathrm{I} - T)^s \quad \to \quad A_n x = o(1/n^s).$$

Starting Point

[Deriennic-Lin 2001]

$0 < s < 1, \quad (1-z)^s = \sum_{n=0}^{\infty} a_n^{(s)} z^n, \quad \to \quad a^{(s)} \in \ell^1$

$(\mathrm{I} - T)^s := \sum_{n=0}^{\infty} a_n^{(s)} T^n$

$$x \in \operatorname{ran}(\mathrm{I} - T)^s \quad \to \quad A_n x = o(1/n^s).$$

$\log(\mathrm{I} - T)$ defined as generator of semigroup $((\mathrm{I} - T)^s)_{s>0}$.

1. uniform rate for $x \in \operatorname{dom}(\log(\mathrm{I} - T))$?

Starting Point

[Deriennic-Lin 2001]

$0 < s < 1, \quad (1-z)^s = \sum_{n=0}^{\infty} a_n^{(s)} z^n, \quad \to \quad a^{(s)} \in \ell^1$

$(\mathrm{I} - T)^s := \sum_{n=0}^{\infty} a_n^{(s)} T^n$

$$x \in \mathrm{ran}(\mathrm{I} - T)^s \quad \to \quad A_n x = o(1/n^s).$$

$\log(\mathrm{I} - T)$ defined as generator of semigroup $((\mathrm{I} - T)^s)_{s>0}$.

1. uniform rate for $x \in \mathrm{dom}(\log(\mathrm{I} - T))$?
2. is $x \in \mathrm{dom}(\log(\mathrm{I} - T))$ equivalent to

$$H_T x := -\sum_{n=1}^{\infty} \frac{T^n x}{n}$$

being convergent?

Starting Point

[Deriennic-Lin 2001]

$0 < s < 1, \quad (1-z)^s = \sum_{n=0}^{\infty} a_n^{(s)} z^n, \quad \rightarrow \quad a^{(s)} \in \ell^1$

$(\mathrm{I} - T)^s := \sum_{n=0}^{\infty} a_n^{(s)} T^n$

$$x \in \mathrm{ran}(\mathrm{I} - T)^s \quad \rightarrow \quad A_n x = o(1/n^s).$$

$\log(\mathrm{I} - T)$ defined as generator of semigroup $((\mathrm{I} - T)^s)_{s>0}$.

1. uniform rate for $x \in \mathrm{dom}(\log(\mathrm{I} - T))$?
2. is $x \in \mathrm{dom}(\log(\mathrm{I} - T))$ equivalent to

$$H_T x := -\sum_{n=1}^{\infty} \frac{T^n x}{n}$$

being convergent?

[Assani, Cohen, Cuny, Lin 2003–2009] : yes to (2)

Generalisation

$\alpha = (\alpha_n)_{n \geq 0} \subseteq \mathbb{C}, \qquad f(z) := \widehat{\alpha}(z) = \sum_{n=0}^{\infty} \alpha_n z^n.$

$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}, \qquad f : \mathbb{D} \longrightarrow \mathbb{C}$ holomorphic

Generalisation

$\alpha = (\alpha_n)_{n\geq 0} \subseteq \mathbb{C}, \qquad f(z) := \widehat{\alpha}(z) = \sum_{n=0}^{\infty} \alpha_n z^n.$

$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}, \qquad f : \mathbb{D} \longrightarrow \mathbb{C}$ holomorphic

1. uniform rate for $x \in \mathrm{dom}(f(T))$?

Generalisation

$\alpha = (\alpha_n)_{n\geq 0} \subseteq \mathbb{C}, \qquad f(z) := \widehat{\alpha}(z) = \sum_{n=0}^{\infty} \alpha_n z^n.$

$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}, \qquad f : \mathbb{D} \longrightarrow \mathbb{C}$ holomorphic

1. uniform rate for $x \in \mathrm{dom}(f(T))$?
2. is $x \in \mathrm{dom}(\log(\mathrm{I} - T))$ equivalent to

$$\sum_{n=0}^{\infty} \alpha_n T^n x$$

being convergent?

Generalisation

$\alpha = (\alpha_n)_{n \geq 0} \subseteq \mathbb{C}, \qquad f(z) := \widehat{\alpha}(z) = \sum_{n=0}^{\infty} \alpha_n z^n.$

$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}, \qquad f : \mathbb{D} \longrightarrow \mathbb{C}$ holomorphic

1. uniform rate for $x \in \operatorname{dom}(f(T))$?
2. is $x \in \operatorname{dom}(\log(\mathrm{I} - T))$ equivalent to

   $$\sum_{n=0}^{\infty} \alpha_n T^n x$$

   being convergent?

$\longrightarrow$ needs a functional calculus to be meaningful

Functional Calculus

$\mathrm{A}^1_+(\mathbb{D}) := \{\widehat{\alpha} \mid \alpha \in \ell^1\} \qquad \|\widehat{\alpha}\|_{\mathrm{A}^1_+} := \|\alpha\|_1$

Functional Calculus

$\mathrm{A}^1_+(\mathbb{D}) := \{\widehat{\alpha} \mid \alpha \in \ell^1\} \qquad \|\widehat{\alpha}\|_{\mathrm{A}^1_+} := \|\alpha\|_1$

$\widehat{\alpha}(T) := \sum_{j=0}^{\infty} \alpha_j T^j$

$$(f \longmapsto f(T)) : \mathrm{A}^1_+(\mathbb{D}) \longrightarrow \mathcal{L}(X)$$

algebra homomorphism, $\quad \|f(T)\| \leq M \|f\|_{\mathrm{A}^1_+}$.

Functional Calculus

$\mathrm{A}^1_+(\mathbb{D}) := \{\widehat{\alpha} \mid \alpha \in \ell^1\} \qquad \|\widehat{\alpha}\|_{\mathrm{A}^1_+} := \|\alpha\|_1$

$\widehat{\alpha}(T) := \sum_{j=0}^{\infty} \alpha_j T^j$

$$(f \longmapsto f(T)) : \mathrm{A}^1_+(\mathbb{D}) \longrightarrow \mathcal{L}(X)$$

algebra homomorphism, $\quad \|f(T)\| \leq M \, \|f\|_{\mathrm{A}^1_+}$.

$\longrightarrow$ canonical extension towards unbounded functions/operators via regularisation

Functional Calculus

$\mathrm{A}^1_+(\mathbb{D}) := \{\widehat{\alpha} \mid \alpha \in \ell^1\} \qquad \|\widehat{\alpha}\|_{\mathrm{A}^1_+} := \|\alpha\|_1$

$\widehat{\alpha}(T) := \sum_{j=0}^{\infty} \alpha_j T^j$

$$(f \longmapsto f(T)) : \mathrm{A}^1_+(\mathbb{D}) \longrightarrow \mathcal{L}(X)$$

algebra homomorphism, $\quad \|f(T)\| \leq M\, \|f\|_{\mathrm{A}^1_+}$.

$\longrightarrow$ canonical extension towards unbounded functions/operators via regularisation

If $\quad f = 1/g \quad$ and $\quad g \in \mathrm{A}^1_+(\mathbb{D}) \quad$ s.t. $\quad g(T) \quad$ is injective, then

$$f(T) = g(T)^{-1}.$$

In this case $\quad \mathrm{dom}(f(T)) = \mathrm{ran}(g(T))$.

Renewal Sequences

It turns out that the questions become tractable in the case that $g = 1/f \in \mathrm{A}^1_+(\mathbb{D})$ has a special form.

Renewal Sequences

It turns out that the questions become tractable in the case that $g = 1/f \in \mathrm{A}^1_+(\mathbb{D})$ has a special form.

Definition: α is a *renewal sequence* if there is $\gamma = (\gamma_j)_{j\geq 1}$ s.t.

$$1 = \widehat{\alpha}(z)(1 - \sum_{j=1}^{\infty} \gamma_j z^j) \qquad (z \in \mathbb{D})$$

with $\gamma_j \geq 0$ for all $j \geq 1$ and $\sum_{j=1}^{\infty} \gamma_j \leq 1$.

Renewal Sequences

It turns out that the questions become tractable in the case that $g = 1/f \in \mathrm{A}^1_+(\mathbb{D})$ has a special form.

Definition: α is a *renewal sequence* if there is $\gamma = (\gamma_j)_{j\geq 1}$ s.t.

$$1 = \widehat{\alpha}(z)(1 - \sum_{j=1}^{\infty} \gamma_j z^j) \qquad (z \in \mathbb{D})$$

with $\gamma_j \geq 0$ for all $j \geq 1$ and $\sum_{j=1}^{\infty} \gamma_j \leq 1$.

$$f = \widehat{\alpha}, \quad g = 1/f, \quad g(z) = 1 - \sum_{j=1}^{\infty} \gamma_j z^j$$

$\longrightarrow \quad \alpha_0 = 1,\ 0 \leq \alpha \leq \mathbf{1}$

Renewal Sequences

It turns out that the questions become tractable in the case that $g = 1/f \in \mathrm{A}^1_+(\mathbb{D})$ has a special form.

Definition: α is a *renewal sequence* if there is $\gamma = (\gamma_j)_{j\geq 1}$ s.t.

$$1 = \widehat{\alpha}(z)(1 - \sum_{j=1}^{\infty} \gamma_j z^j) \qquad (z \in \mathbb{D})$$

with $\gamma_j \geq 0$ for all $j \geq 1$ and $\sum_{j=1}^{\infty} \gamma_j \leq 1$.

$$f = \widehat{\alpha}, \quad g = 1/f, \quad g(z) = 1 - \sum_{j=1}^{\infty} \gamma_j z^j$$

$\longrightarrow \quad \alpha_0 = 1,\ 0 \leq \alpha \leq \mathbf{1}$

Interesting case: $\sum_{j=1}^{\infty} \gamma_j = 1$ (i.e., $f(1) = \infty$)

Relevant Examples

Relevant Examples

negative fractional powers: $\qquad 0 < s < 1, \quad f(z) = (1-z)^{-s}.$

$$g(z) = (1/f)(z) = (1-z)^s = \sum_{n=1}^{\infty} a_n^{(s)} z^n$$
$$= 1 - \sum_{j=1}^{\infty} \frac{s(1-s)\cdot \cdots \cdot (j-1-s)}{n!} z^j.$$

Relevant Examples

negative fractional powers: $0 < s < 1, \quad f(z) = (1-z)^{-s}.$

$$g(z) = (1/f)(z) = (1-z)^s = \sum_{n=1}^{\infty} a_n^{(s)} z^n$$

$$= 1 - \sum_{j=1}^{\infty} \frac{s(1-s)\cdot \cdots \cdot (j-1-s)}{n!} z^j.$$

modified logarithm: $f(z) = \frac{-\log(1-z)}{z} = \sum_{n=0}^{\infty} \frac{1}{n+1} z^n$

Relevant Examples

negative fractional powers: $0 < s < 1, \quad f(z) = (1-z)^{-s}$.

$$g(z) = (1/f)(z) = (1-z)^s = \sum_{n=1}^{\infty} a_n^{(s)} z^n$$

$$= 1 - \sum_{j=1}^{\infty} \frac{s(1-s)\cdot \cdots \cdot (j-1-s)}{n!} z^j.$$

modified logarithm: $f(z) = \frac{-\log(1-z)}{z} = \sum_{n=0}^{\infty} \frac{1}{n+1} z^n$

Theorem (Kaluza): If $\alpha \geq 0$ is bounded and satisfies

$$\alpha_0 = 1, \quad \alpha_k^2 \leq \alpha_{k-1}\alpha_{k+1} \qquad (k \geq 1)$$

then α is a renewal sequence (and is decreasing).

Domain Characterisation (1)

α renewal sequence, $f = \widehat{\alpha}$, $g = 1/f = 1 - \sum_{j=1}^{\infty} \gamma_j z^j$.

Domain Characterisation (1)

α renewal sequence, $f = \widehat{\alpha}$, $g = 1/f = 1 - \sum_{j=1}^{\infty} \gamma_j z^j$.

in addition: $\alpha \in \mathrm{BV}$, i.e., $(1 - z)f(z) \in \mathrm{A}^1_+(\mathbb{D})$.

$\longrightarrow$ $f(T)$ def'd by the fc as $f(T) = (\mathrm{I} - T)^{-1}[(1 - z)f](T)$.

Domain Characterisation (1)

α renewal sequence, $f = \widehat{\alpha}$, $g = 1/f = 1 - \sum_{j=1}^{\infty} \gamma_j z^j$.

in addition: $\alpha \in \mathrm{BV}$, i.e., $(1-z)f(z) \in \mathrm{A}^1_+(\mathbb{D})$.

$\longrightarrow$ $f(T)$ def'd by the fc as $f(T) = (\mathrm{I} - T)^{-1}[(1-z)f](T)$.

$x \in \mathrm{dom}(f(T)) \overset{?}{\iff} \sum_{n=0}^{\infty} \alpha_k T^k x$ converges (in some sense).

Domain Characterisation (1)

α renewal sequence, $f = \widehat{\alpha}$, $g = 1/f = 1 - \sum_{j=1}^{\infty} \gamma_j z^j$.

in addition: $\alpha \in \mathrm{BV}$, i.e., $(1-z)f(z) \in \mathrm{A}^1_+(\mathbb{D})$.

$\longrightarrow$ $f(T)$ def'd by the fc as $f(T) = (\mathrm{I} - T)^{-1}[(1-z)f](T)$.

$x \in \mathrm{dom}(f(T)) \quad \overset{?}{\Longleftrightarrow} \quad \sum_{n=0}^{\infty} \alpha_k T^k x$ converges (in some sense).

Example: $f(z) = -\log(1-z)/z = \sum_{k=0}^{\infty} z^k/(k+1)$

$$\longrightarrow \quad h(z) = (1-z)f(z) = 1 - \sum_{k=1}^{\infty} \frac{z^k}{k(k+1)} \in \mathrm{A}^1_+(\mathbb{D}),$$

Domain Characterisation (1)

α renewal sequence, $f = \widehat{\alpha}$, $g = 1/f = 1 - \sum_{j=1}^{\infty} \gamma_j z^j$.

in addition: $\alpha \in \mathrm{BV}$, i.e., $(1-z)f(z) \in \mathrm{A}_+^1(\mathbb{D})$.

$\longrightarrow$ $f(T)$ def'd by the fc as $f(T) = (\mathrm{I} - T)^{-1}[(1-z)f](T)$.

$x \in \mathrm{dom}(f(T)) \quad \overset{?}{\Longleftrightarrow} \quad \sum_{n=0}^{\infty} \alpha_k T^k x$ converges (in some sense).

Example: $f(z) = -\log(1-z)/z = \sum_{k=0}^{\infty} z^k/(k+1)$

$$\longrightarrow \quad h(z) = (1-z)f(z) = 1 - \sum_{k=1}^{\infty} \frac{z^k}{k(k+1)} \in \mathrm{A}_+^1(\mathbb{D}),$$

- $\log(1-z) = f - h$, $\mathrm{dom}(f(T)) = \mathrm{dom}(\log(\mathrm{I} - T))$.
- $H_T x = \sum_{k=1}^{\infty} \frac{-1}{k} T^k x$ conv. iff $\sum_{k=0}^{\infty} \frac{1}{k+1} T^k x$ conv.

Domain Characterisation (2)

Domain Characterisation (2)

"$\Leftarrow$": always true (easy)

Domain Characterisation (2)

"$\Leftarrow$": always true (easy)

"$\Rightarrow$": true if $\alpha_n T^{n+1} g(T) \to 0$ strongly:

Domain Characterisation (2)

"$\Leftarrow$": always true (easy)

"$\Rightarrow$": true if $\alpha_n T^{n+1} g(T) \to 0$ strongly:

Theorem: α renewal sequence, $n \in \mathbb{N}$. Then there is a renewal sequence $\beta = \beta(n)$ such that

$$\widehat{\alpha}(z) = \left(\sum_{j=0}^{n} \alpha_j z^j \right) \cdot \widehat{\beta}(z)$$

$$g_n := \sum_{j=0}^{n} \alpha_j z^j / f = 1/\widehat{\beta} = 1 - \sum_{k=n}^{\infty} \gamma_{k,n} z^k$$

Domain Characterisation (2)

"$\Leftarrow$": always true (easy)

"$\Rightarrow$": true if $\alpha_n T^{n+1} g(T) \to 0$ strongly:

Theorem: α renewal sequence, $n \in \mathbb{N}$. Then there is a renewal sequence $\beta = \beta(n)$ such that

$$\widehat{\alpha}(z) = \left(\sum_{j=0}^{n} \alpha_j z^j \right) \cdot \widehat{\beta}(z)$$

$$g_n := \sum_{j=0}^{n} \alpha_j z^j / f = 1/\widehat{\beta} = 1 - \sum_{k=n}^{\infty} \gamma_{k,n} z^k$$

$\longrightarrow$ $\sup_n \|g_n\|_{\mathrm{A}^1_+} \leq 2$ $(= 2, \text{ if } f(1) = \infty)$

Domain Characterisation (2)

"$\Leftarrow$": always true (easy)

"$\Rightarrow$": true if $\alpha_n T^{n+1} g(T) \to 0$ strongly:

Theorem: α renewal sequence, $n \in \mathbb{N}$. Then there is a renewal sequence $\beta = \beta(n)$ such that

$$\widehat{\alpha}(z) = \Big(\sum\nolimits_{j=0}^{n} \alpha_j z^j \Big) \cdot \widehat{\beta}(z)$$

$$g_n := \sum\nolimits_{j=0}^{n} \alpha_j z^j / f = 1/\widehat{\beta} = 1 - \sum\nolimits_{k=n}^{\infty} \gamma_{k,n} z^k$$

$\longrightarrow$ $\sup_n \|g_n\|_{\mathrm{A}^1_+} \le 2$ $\quad (= 2, \text{ if } f(1) = \infty)$

$\longrightarrow$ $g_n(T) \to \mathrm{I}$ strongly $\quad (\operatorname{ran}(\mathrm{I} - T)$ dense in $X!)$

Rates of Convergence

Rates of Convergence

$g(z) = 1 - \sum_{j=1}^{\infty} \gamma_j z^j, \quad \sum_{j=1}^{\infty} \gamma_j = 1, \quad \gamma_j \geq 1.$

Rates of Convergence

$$g(z) = 1 - \sum_{j=1}^{\infty} \gamma_j z^j, \quad \sum_{j=1}^{\infty} \gamma_j = 1, \quad \gamma_j \geq 1.$$

$$\begin{aligned} g_n(z) &:= \Big(\sum_{k=0}^{n-1} z^k \Big) g(z) = \dots \\ &= 1 + (1-\gamma_1)z - (1-\gamma_1-\gamma_2)z^2 + \dots + (1-\gamma_1-\dots-\gamma_{n-1})z^{n-1} \\ &\quad - \sum_{k=n}^{\infty} (\gamma_{k-n+1} + \dots + \gamma_k) z^n \end{aligned}$$

Rates of Convergence

$$g(z) = 1 - \sum_{j=1}^{\infty} \gamma_j z^j, \quad \sum_{j=1}^{\infty} \gamma_j = 1, \quad \gamma_j \geq 1.$$

$$g_n(z) := \Big(\sum_{k=0}^{n-1} z^k \Big) g(z) = \dots$$

$$= 1 + (1-\gamma_1)z - (1-\gamma_1-\gamma_2)z^2 + \dots + (1-\gamma_1-\dots-\gamma_{n-1})z^{n-1}$$

$$- \sum_{k=n}^{\infty} (\gamma_{k-n+1} + \dots + \gamma_k) z^n$$

$$\longrightarrow \qquad \|A_n g(T)x\| \leq 2M \|g_n\|_{\mathrm{A}_+^1} \|x\| = 2M r_n \|x\|$$

$$r_n := \left(\sum_{j>n} \gamma_j + \frac{1}{n} \sum_{j=1}^{n} j\gamma_j \right)$$

$$\longrightarrow \quad x \in \operatorname{ran}(g(T)) \ \to \ A_n x = O(r_n).$$

Rates of Convergence

$g(z) = 1 - \sum_{j=1}^{\infty} \gamma_j z^j, \quad \sum_{j=1}^{\infty} \gamma_j = 1, \quad \gamma_j \geq 1.$

$$g_n(z) := \Big(\sum_{k=0}^{n-1} z^k \Big) g(z) = \dots$$
$$= 1 + (1-\gamma_1)z - (1-\gamma_1-\gamma_2)z^2 + \cdots + (1-\gamma_1-\cdots-\gamma_{n-1})z^{n-1}$$
$$- \sum_{k=n}^{\infty} (\gamma_{k-n+1} + \cdots + \gamma_k) z^n$$

$\longrightarrow \qquad \|A_n g(T)x\| \leq 2M \|g_n\|_{\mathrm{A}^1_+} \|x\| = 2M r_n \|x\|$

$$r_n := \left(\sum_{j>n} \gamma_j + \frac{1}{n} \sum_{j=1}^{n} j\gamma_j \right)$$

$\longrightarrow \quad x \in \operatorname{ran}(g(T)) \ \to \ A_n x = O(r_n).$

Q: Which convergence rates can be realised?

Logarithmic rate

Logarithmic rate

$$f(z) = \frac{-\log(1-z)}{z} = \sum_{k=0}^{\infty} \frac{1}{k+1} z^k, \quad g = f^{-1} = 1 - \sum_{j=1}^{\infty} \gamma_j z^j$$

Logarithmic rate

$$f(z) = \frac{-\log(1-z)}{z} = \sum_{k=0}^{\infty} \frac{1}{k+1} z^k, \quad g = f^{-1} = 1 - \sum_{j=1}^{\infty} \gamma_j z^j$$

Lemma: $$\gamma_k = \frac{1}{k(\log k)^2} + O\left(\frac{1}{k(\log k)^3}\right)$$

Logarithmic rate

$$f(z) = \frac{-\log(1-z)}{z} = \sum_{k=0}^{\infty} \frac{1}{k+1} z^k, \qquad g = f^{-1} = 1 - \sum_{j=1}^{\infty} \gamma_j z^j$$

Lemma: $$\gamma_k = \frac{1}{k(\log k)^2} + O\left(\frac{1}{k(\log k)^3}\right)$$

$\longrightarrow \quad r_n \sim 1/\log n$

Logarithmic rate

$$f(z) = \frac{-\log(1-z)}{z} = \sum_{k=0}^{\infty} \frac{1}{k+1} z^k, \qquad g = f^{-1} = 1 - \sum_{j=1}^{\infty} \gamma_j z^j$$

Lemma: $$\gamma_k = \frac{1}{k(\log k)^2} + O\left(\frac{1}{k(\log k)^3}\right)$$

$\longrightarrow \quad r_n \sim 1/\log n$

- best possible general result (prototypical instance + PUB)

Logarithmic rate

$f(z) = \frac{-\log(1-z)}{z} = \sum_{k=0}^{\infty} \frac{1}{k+1} z^k, \quad g = f^{-1} = 1 - \sum_{j=1}^{\infty} \gamma_j z^j$

Lemma: $\gamma_k = \dfrac{1}{k(\log k)^2} + O\left(\dfrac{1}{k(\log k)^3}\right)$

$\longrightarrow \quad r_n \sim 1/\log n$

- best possible general result (prototypical instance + PUB)
- $1 \in \sigma(T) \;\rightarrow\; \exists x \in \mathrm{ran}(g(T))$ without polyn. rate

Logarithmic rate

$$f(z) = \frac{-\log(1-z)}{z} = \sum_{k=0}^{\infty} \frac{1}{k+1} z^k, \quad g = f^{-1} = 1 - \sum_{j=1}^{\infty} \gamma_j z^j$$

Lemma: $$\gamma_k = \frac{1}{k(\log k)^2} + O\left(\frac{1}{k(\log k)^3}\right)$$

$\longrightarrow \quad r_n \sim 1/\log n$

- best possible general result (prototypical instance + PUB)
- $1 \in \sigma(T) \;\rightarrow\; \exists x \in \operatorname{ran}(g(T))$ without polyn. rate
- not sufficient: $A_n x = O(1/\log n) \;\not\rightarrow\; x \in \operatorname{ran}(g(T))$ [Cohen-Lin 2009]

References

- M.Haase and Y. Tomilov, *Domain characterizations of certain functions of power-bounded operators*, submitted.
- M.Haase, A.M. Gomilko, Y. Tomilov, *Non-polynomial rates in ergodic theorems*, in preparation.

References

- M.Haase and Y. Tomilov, *Domain characterizations of certain functions of power-bounded operators*, submitted.
- M.Haase, A.M. Gomilko, Y. Tomilov, *Non-polynomial rates in ergodic theorems*, in preparation.
- Y. Derriennic and M. Lin, *Fractional Poisson equations and ergodic theorems for fractional coboundaries*, Israel J. Math. 123 (2001), 93-130.
- I. Assani and N. Lin, *The one-sided ergodic Hilbert transform*, Comtemp. Math. 430 (2007), 21–39.
- G. Cohen and M. Lin, *The one-sided ergodic Hilbert transform of normal contractions*, to appear 2009.
- G. Cohen, C. Cuny and M. Lin, *The one-sided ergodic Hilbert transform on Banach spaces*, preprint.

References

- M.Haase and Y. Tomilov, *Domain characterizations of certain functions of power-bounded operators*, submitted.
- M.Haase, A.M. Gomilko, Y. Tomilov, *Non-polynomial rates in ergodic theorems*, in preparation.
- Y. Deriennic and M. Lin, *Fractional Poisson equations and ergodic theorems for fractional coboundaries*, Israel J. Math. 123 (2001), 93-130.
- I. Assani and N. Lin, *The one-sided ergodic Hilbert transform*, Comtemp. Math. 430 (2007), 21–39.
- G. Cohen and M. Lin, *The one-sided ergodic Hilbert transform of normal contractions*, to appear 2009.
- G. Cohen, C. Cuny and M. Lin, *The one-sided ergodic Hilbert transform on Banach spaces*, preprint.
- M. Haase, *The functional calculus for sectorial operators*, Birkhäuser Basel, 2006.