Analysis of Fluid-Structure Interactive PDE Models

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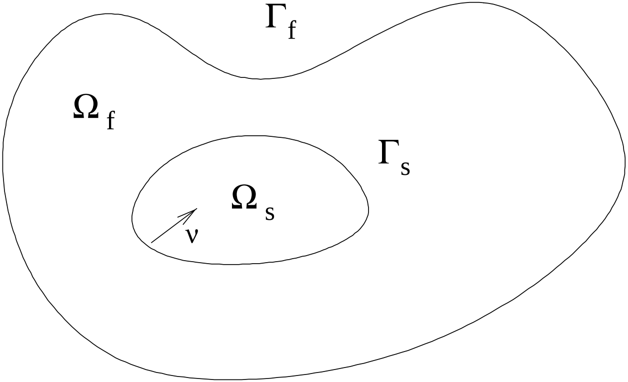
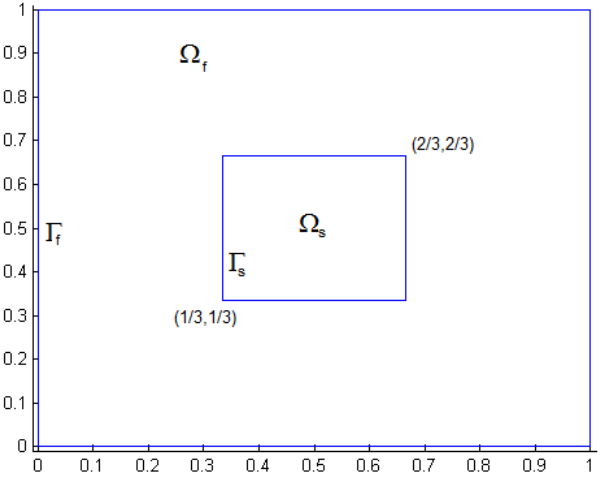
The Fluid-Structure Domain

A General Bounded Domain



The Fluid-Structure Domain

A Canonical Example



The Modeling PDE

Let $u = [u_1, u_2, ..., u_n]^T$ be a n-dimensional ($n = 2, 3$) velocity field, $p(t, x)$ a scalar-valued pressure, and $w = [w_1, w_2, ..., w_n]^T$ a displacement field for the solid.

$$\text{(PDE)} \quad \begin{cases} u_t - \Delta u + \nabla p = 0 \ \text{ in } (0, T) \times \Omega_f \\ div(u) = 0 \ \text{ in } (0, T) \times \Omega_f \\ w_{tt} - \Delta w + w = 0 \ \text{ in } (0, T) \times \Omega_s \end{cases}$$

$$\text{(BC)} \quad \begin{cases} u|_{\Gamma_f} = 0 \\ u = w_t \ \text{on } (0, T) \times \Gamma_s \\ \frac{\partial u}{\partial \nu} - \frac{\partial w}{\partial \nu} = p\nu \ \text{ on } (0, T) \times \Gamma_s \end{cases}$$

$$\text{(IC)} \quad [w(0, \cdot), w_t(0, \cdot), u(0, \cdot)]^T = [w_0, w_1, u_0]^T \in \mathbf{H},$$

$$\begin{aligned} \text{where} \quad \mathbf{H} &\equiv [H^1(\Omega_s)]^n \times [L^2(\Omega_s)]^n \times \mathcal{H}_f, \text{ and} \\ \mathcal{H}_f &= \{ f \in [L^2(\Omega_f)]^n : div(f) = 0 \text{ in } \Omega_f \\ &\qquad \text{and } [f \cdot \nu]_{\Gamma_f} = 0 \} . \end{aligned}$$

The appearance here is for ease of explication. In the literature, the PDE is a little more involved:

$$\text{PDEs} \quad \begin{cases} u_t - \nabla \cdot \left(\nabla u + \nabla u^T \right) + \nabla p = 0 \quad \text{in } (0, T) \times \Omega_f \\ div(u) = 0 \quad \text{in } (0, T) \times \Omega_f \\ w_{tt} - div(\sigma(w)) + w = 0 \quad \text{in } (0, T) \times \Omega_s \end{cases}$$

$$\text{B.C.} \quad \begin{cases} \left(\nabla u + \nabla u^T \right) \cdot \nu = \sigma(w) \cdot \nu + p\nu \quad \text{on } (0, T) \times \Gamma_s \\ u|_{\Gamma_f} = 0 \quad \text{on } (0, T) \times \Gamma_f \\ w_t|_{\Gamma_s} = u|_{\Gamma_s} \quad \text{on } (0, T) \times \Gamma_s, \end{cases}$$

with the same initial conditions

$$\text{I.C.} \quad [w(0), w_t(0), u(0)] = [w_0, w_1, u_0] \in \mathbf{H}.$$

Here $\sigma(\cdot)$ is the stress tensor from the Lamé system of elasticity.

A Very Brief History of the PDE

This "fluid-structure" PDE was originally proposed in [J. Lions, 1967] and subsequently in [Q. Du, M.D. Gunzburger, L.S. Hou, J. Lee, 2003] et al., the latter paper also providing an interesting review of the literature, as regards the various classes of such interactive PDE models. We defer to that reference for the history and development of these physically relevant PDE's. which are currently invoked to describe the coupling of fluid and solid; but by way of emphasizing the novelty of the fluid-structure problem under present consideration, we explicitly quote here the following statement from the paper:

> *The majority of the references cited use solid models in lower spatial dimensions, e.g., one-dimensional beams interacting with two-dimensional fluids or two-dimensional plates interacting with three-dimensional fluids. Rigorous mathematical results are rare for fluid-structure interaction problems in which both the fluid and the solid occupy true spatial domains.*

There is also the recent and ongoing work of [V. Barbu, Z. Grujić, I. Lasiecka and A. Tuffaha, 2007], the authors also provide a wellposedness and regularity theory for the fluid-structure PDE, for linear and nonlinear versions of the model. The methodology is wholly different than that used to obtain the results posted in the present paper. In particular: As we shall see, the elimination of the pressure term cannot be accomplished by an application of the classic Leray (or Helmholtz) Projector, as is typically done with uncoupled fluid flow PDE models under the so-called "no-slip" boundary condition; the situation calls for a different approach.

Elimination of the Pressure

- In classical (uncoupled) Stokes theory, pressure is eliminated by means of the Leray projector, but such approach is invalid here.
- Rather, in the present problem pressure is eliminated by its identification as a solution of a certain BVP.

In fact, one can verify directly that pressure variable $p(t)$ satisfies the following elliptic problem:

$$\begin{cases} \Delta p = 0 \ \text{ in } \Omega_f \\ p = \frac{\partial u}{\partial \nu} \cdot \nu - \frac{\partial w}{\partial \nu} \cdot \nu \ \text{ on } \Gamma_s \\ \frac{\partial p}{\partial \nu} = \Delta u \cdot \nu \ \text{ on } \Gamma_f . \end{cases}$$

As such, the pressure then admits of the representation,

$$\begin{aligned} p(t) &= D_s\left[\left(\frac{\partial u(t)}{\partial \nu}\cdot\nu - \frac{\partial w(t)}{\partial \nu}\cdot\nu\right)_{\Gamma_s}\right] \\ &\qquad + N_f\left[(\Delta u(t)\cdot\nu)_{\Gamma_f}\right] \text{ in } \Omega_f; \end{aligned}$$

where (i) "Dirichlet" map D_s is defined by

$$h = D_s(g) \Longleftrightarrow \begin{cases} \Delta h = 0 & \text{in } \Omega_f \\ h = g & \text{on } \Gamma_s \\ \frac{\partial h}{\partial \nu} = 0 & \text{on } \Gamma_f; \end{cases}$$

and (ii) "Neumann" map N_f is the Neumann map defined by

$$h = N_f(g) \Longleftrightarrow \begin{cases} \Delta h = 0 & \text{in } \Omega_f \\ h = 0 & \text{on } \Gamma_s \\ \frac{\partial h}{\partial \nu} = g & \text{on } \Gamma_f. \end{cases}$$

Upon substitution, the fluid-structure system thus becomes

$$\begin{cases} u_t - \Delta u + G_1 w + G_2 u = 0 \text{ in } (0, T) \times \Omega_f \\ div(u) = 0 \text{ in } (0, T) \times \Omega_f \\ w_{tt} - \Delta w + w = 0 \text{ in } (0, T) \times \Omega_s \end{cases}$$

$$\begin{cases} u|_{\Gamma_f} = 0 \\ u = w_t \text{ on } (0, T) \times \Gamma_s \end{cases}$$

where

$$\begin{aligned} G_1 w &\equiv \nabla \left(D_s \left[\left(\frac{\partial w}{\partial \nu} \cdot \nu \right)_{\Gamma_s} \right] \right) \\ G_2 u &\equiv -\nabla \left(D_s \left[\left(\frac{\partial u}{\partial \nu} \cdot \nu \right)_{\Gamma_s} \right] + N_f [(\Delta u \cdot \nu)_{\Gamma_f}] \right). \end{aligned}$$

Abstract Model

With these "Green's maps" in mind, we can then define the following matrix $\mathbf{A} : D(\mathbf{A}) \subset \mathbf{H} \to \mathbf{H}$ as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & I & 0 \\ \Delta - I & 0 & 0 \\ G_1 & 0 & \Delta + G_2 \end{bmatrix}.$$

Via this operator, the fluid-structure interaction may then be written as the abstract Cauchy problem,

$$\frac{d}{dt}\begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \mathbf{A}\begin{bmatrix} w \\ w_t \\ u \end{bmatrix}; \quad \begin{bmatrix} w(0) \\ w_t(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} u_0 \\ w_0 \\ w_1 \end{bmatrix}.$$

Theorem

([G.A., R. Triggiani, 2006],[G.A. R. Triggiani, 2009], [G.A., M. Dvorak, 2008]). The FS-generator $\mathbf{A}$ *is maximal dissipative when* $D(\mathbf{A}) \subset \mathbf{H}$ *is taken to be the following subspace:* $D(\mathbf{A})$ *comprises all triples* $[w_0, w_1, u_0] \in \mathbf{H}$ *for which there exists an associated "pressure" function* $\pi_0 = \pi_0(w_0, u_0) \in [L^2(\Omega_f)]^n$ *such that* $[w_0, w_1, u_0, \pi_0]$ *collectively satisfy the following properties:*

(i) $w_0 \in [H^1(\Omega_s)]^n$, with $\Delta w_0 \in [L^2(\Omega_s)]^n$

(and so $\left.\dfrac{\partial w_0}{\partial \nu}\right|_{\Gamma_s} \in [H^{-\frac{1}{2}}(\Gamma_s)]^n$);

(ii) $w_1 \in [H^1(\Omega_s)]^n$;

(iii) $u_0 \in \mathcal{H}_f \cap [H^1(\Omega_f)]^n$, with $\Delta u_0 - \nabla \pi_0 \in \mathcal{H}_f$;

(iv) $\left.\dfrac{\partial u_0}{\partial \nu}\right|_{\Gamma_s} \in [H^{-\frac{1}{2}}(\Gamma_s)]^n$ and $\pi_0|_{\Gamma_s} \in H^{-\frac{1}{2}}(\Gamma_s)$;

(v) $[(\Delta u_0) \cdot \nu]_{\Gamma_f} \in H^{-\frac{3}{2}}(\Gamma_f)$;

(vi) $\left.\dfrac{\partial u_0}{\partial \nu}\right|_{\Gamma_s} = \left[\dfrac{\partial w_0}{\partial \nu} + \pi_0 \nu\right]_{\Gamma_s}$;

(vii) $u_0|_{\Gamma_f} = 0$ on Γ_f;

(viii) $w_1|_{\Gamma_s} = u_0|_{\Gamma_s}$ on Γ_s, in $[H^{\frac{1}{2}}(\Gamma_s)]^n$.

Theorem

([G.A., R. Triggiani, 2006-2007],[G.A., R. Triggiani, 2009]) If the interface Γ_s is such that no eigenvalues (other than $\lambda = 0$) lie on the imaginary axis - e.g., Γ_s is partially flat - then the fluid structure solutions decay asymptotically on $[Null(\mathbf{A})]^\perp \subset \mathbf{H}$.

Remark: The resolvent $(\lambda - \mathbf{A})^{-1}$ is *not* compact, and so the proof here is not classical. It uses the spectral criterion in [Arendt-Batty/Lyubich-Phong, 1989]; plus either (i) a thorough analysis to eliminate the possibility of continuous spectrum on $i\mathbb{R}$; or (ii) a uniform estimate on $\sqrt{\alpha}\left\|(\alpha + i\beta - \mathbf{A})^{-1}\right\|$, where $\beta \in \mathbb{R} \setminus M$, where M is a certain countable set, and α is positive (and small). (this latter approach inspired by [Boyadzhiev and Levan, 1995], [Tomilov, 2001], and [Chill, Tomilov, 2005]).

Theorem

([G.A., R. Triggiani, 2007], [G.A., R. Triggiani, 2009]). Under appropriate appropriate boundary feedback, the corresponding solutions of the FS feedback system decay exponentially.

Remark. The proof of this result involves, in part a "multiplier method" which to some extent is a vector-valued version of that carried out for boundary-controlled (and scalar-valued) wave equations; see e.g., [R. Triggiani, 1989], which follows the Lyapunov method-based papers [G. Chen, 1981] and [J. Lagnese, 1983]. A key feature of the our uniform decay Theorem is the validity of the decay rate with *no* geometrical assumptions being imposed upon the boundary interface Γ_s. The "big gun" which allows for this generality are available microlocal results, which provides for the treatment of boundary integrals involving the tangential derivative $\partial w/\partial\tau$ (or $\nabla_\tau w$) which appears in the multiplier estimates. (see [I. Lasiecka and R. Triggiani, 1992] and [M.A. Horn, 1998]).

Results of Backwards Uniqueness

Backwards Uniqueness Property. Given Banach space X, let $A : D(A) \subset X \to X$ be a C_0-semigroup. Then $\{e^{At}\}_{t \geq 0} \subset \mathcal{L}(X)$ is said to satisfy the *backward-uniqueness property* (BUP) if,

$$\begin{aligned} \text{whenever } e^{AT_0}x_0 &= 0 \text{ for some } T_0 > 0 \\ \text{and } x_0 &\in X, \text{ then } x_0 = 0. \end{aligned}$$

(This abstract property has implications in establishing approximate controllability.)

Theorem

([G.A., R. Triggiani, 2007], [G.A., R. Triggiani, 2008]). The FS-semigroup obeys the BUP.

Remark: The establishment of the BUP depends upon being able to invoke the following operator theoretic result:

Theorem

(see Theorem 3.1 of [Lasiecka-Triggiani-Renardy, 2001]) Let A be the infinitesimal generator of a C_0-semigroup in a Banach space X. Assume there exists constants $\theta \in (\frac{\pi}{2}, \pi)$, $R > 0$ and C, such that

$$\left\| (A - re^{\pm i\theta} I)^{-1} \right\| \leq C. \tag{1}$$

for all $r \geq R$. Then A obeys the backwards uniqueness property.

In fact, in [G.A., R. Triggiani, 2007], [G.A., R. Triggiani, 2008], we show the following decay rate: For $\lambda = \alpha + i\beta$ satisfying

(i) $\lambda = \alpha + i\beta = |\lambda| \, e^{\pm\vartheta}$, for fixed $\frac{3\pi}{4} < \vartheta < \pi$ (so $\left|\tan(\vartheta)\right| < 1$)
(ii) $|\alpha| \geq 1$ is sufficiently large.

. Then one has the estimate.

$$\left\|(\lambda - \mathbf{A})^{-1}\right\|_{\mathcal{L}(\mathbf{H})} \leq \frac{C}{|\alpha|^{\frac{1}{6}}},$$

where C is independent of λ.

Higher Regularity Properties

Note that that the definition $D(\mathbf{A})$ does not provide for "smooth enough" classical solutions; viz.,

$$\begin{aligned} [u_0, w_0, w_1] &\in D(\mathbf{A}) \Rightarrow \\ \{u, p, w\} &\in C([0,T]; H^1(\Omega_f)^n \times L^2(\Omega_f) \times H^1(\Omega_s)^n) \end{aligned}$$

$$\begin{aligned} [u_0, w_0, w_1] &\in D(\mathbf{A}) \nRightarrow \\ \{u, p, w\} &\in C([0,T]; H^2(\Omega_f)^n \times H^1(\Omega_f) \times H^2(\Omega_s)^n). \end{aligned}$$

Not wholly justified then, are the computations needed to generate the necessary a priori inequalities for establishing stabilization and controllability results.

In this connection, we have,

Theorem

([G.A., I. Lasiecka, R. Triggiani, 2008])

$$\begin{array}{rcl} \textit{Let } [u_0, w_0, w_1] & \in & D(\mathbf{A}) \textit{ and } \; w_0 \in H^2(\Omega_s)^n. \textit{ Then:} \\ u & \in & L^\infty(0, T; [H^2(\Omega_f)]^n); \\ w & \in & L^2(0, T; [H^2(\Omega_s)]^n); \\ p & \in & L^2(0, T; H^1(\Omega_f)). \end{array}$$

Here is a brief synopsis of the proof of this Theorem: The key here is to establish needed Sobolev regularity in the *tangential* and then *normal* directions.

To start: Assume $b(\xi)$ is a vector-valued function which is tangential to $\Gamma_s \cup \Gamma_f$; i.e., $b \cdot \nu = 0$ on $\Gamma_s \cup \Gamma_f$. Then we scale the fluid-structure model by the operator $b \cdot \nabla$: Via the change of variable

$$\tilde{u} = (b \cdot \nabla)\, u,\ \tilde{w} = (b \cdot \nabla)\, w,\ \tilde{p} = (b \cdot \nabla)\, p,$$

we arrive at,

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} + \nabla \tilde{p} = [b \cdot \nabla, \Delta]\, u - [b \cdot \nabla, \nabla]\, p \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0, T) \times \Omega_f \\ div(\tilde{u}) = [div, b \cdot \nabla] \cdot u \ \text{ in } (0, T) \times \Omega_f \\ w_{tt} - \Delta w + w = [b \cdot \nabla, \Delta]\, w \ \text{ in } (0, T) \times \Omega_s \end{cases}$$

$$\begin{cases} \tilde{u}\big|_{\Gamma_f} = 0 \\ \tilde{u} = \tilde{w}_t \text{ on } (0, T) \times \Gamma_s \\ \frac{\partial \tilde{u}}{\partial \nu} - \frac{\partial \tilde{w}}{\partial \nu} = \tilde{p}\nu + div(\nu)(\tilde{w} - \tilde{u}) + p\tilde{\nu} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{on } (0, T) \times \Gamma_s \end{cases}$$

$$\begin{aligned} \tilde{w}(0,\cdot) &= (b\cdot\nabla)w_0,\ \tilde{w}_t(0,\cdot) = (b\cdot\nabla)w_1, \\ \tilde{u}(0,\cdot) &= (b\cdot\nabla)u_0 \end{aligned}$$

The higher-than-energy-level commutator terms are troublesome here, but *only in the tangential direction*. To exploit this fact, we invoke the representation of Δ, ∇, and *div* in terms of (local) tangential and normal coordinates (the Melrose-Sjöstrand coordinates, [M-S, 1978]). For example,

$$\Delta = D_\nu^2 + \rho(\nu,\tau)\cdot D_\tau^2 + \ (l.o.t. \text{ in } D_\tau).$$

The troublesome commutators are thus handled in this way, so as to obtain the higher regularity in the tangential direction.

Here, we will focus on the following static problem: For given $\lambda > 0$, we consider the task of finding, for given $[v_1^*, v_2^*, f^*] \in \mathbf{H}$, a solution $[v_1 v_2, f] \in D(\mathbf{A})$ of the equation

$$(\lambda I - \mathbf{A}) \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix},$$

where again

$$\mathbf{A} = \begin{bmatrix} 0 & I & 0 \\ \Delta - I & 0 & 0 \\ G_1 & 0 & \Delta + G_2 \end{bmatrix}.$$

It has been shown ([G. A. and M. Dvorak, 2007-2008],[G. A. and R. Triggiani, 2009]) that the fluid component and then the solid component of the system can be resolved in this way:

If

$$\mathbf{H}^1_{\Gamma_f}(\Omega_f) = \left\{ \phi \in [H^1(\Omega_f)]^n : \phi|_{\Gamma_f} = 0 \right\},$$

then f is the fluid component of the static PDE if and only if $[f, \pi] \in \mathbf{H}^1_{\Gamma_f,0}(\Omega_f) \times L^2(\Omega_f)$ solves the coupled variational relation,

$$\begin{aligned} a_\lambda(f, \phi) + b(\phi, \pi) &= \mathbf{F}(\phi) \text{ for all } \phi \in \mathbf{H}^1_{\Gamma_f,0}(\Omega_f) \\ b(f, \rho) &= 0 \text{ for all } \rho \in L^2(\Omega_f). \end{aligned}$$

Here,

$$
\begin{array}{c}
a_\lambda(\psi,\phi) = \lambda(\psi,\phi)_{\Omega_f} + (\nabla\psi,\nabla\phi)_{\Omega_f} \\
+\frac{1}{\lambda}(\nabla D_\lambda(\psi|_{\Gamma_s}),\nabla D_\lambda(\phi|_{\Gamma_s}))_{\Omega_s} \\
+\frac{(\lambda^2+1)}{\lambda}(D_\lambda(\psi|_{\Gamma_s}),D_\lambda(\phi|_{\Gamma_s}))_{\Omega_s}, \\
\text{for all } \psi,\phi \in \mathbf{H}^1_{\Gamma_f}(\Omega_f);
\end{array}
$$

$$
\begin{array}{rcl}
b(\phi,\rho) & \equiv & -(\rho, div(\phi))_{\Omega_f}, \\
\text{for all } \phi & \in & \mathbf{H}^1_{\Gamma_f}(\Omega_f) \text{ and } \rho \in L^2(\Omega_f);
\end{array}
$$

$$
\begin{array}{l}
\mathbf{F}(\phi) \equiv (f^*,\phi)_{\Omega_f} + (v_2^* + \lambda v_1^*, D_\lambda(\phi|_{\Gamma_s}))_{\Omega_s} \\
-(\lambda^2+1)\left(\frac{1}{\lambda}D_\lambda(v_1^*|_{\Gamma_s}) + \mathbb{A}_\lambda^{-1}(v_2^*+\lambda v_1^*), D_\lambda(\phi|_{\Gamma_s})\right)_{\Omega_s} \\
-\left(\nabla D_\lambda(\frac{1}{\lambda}v_1^*|_{\Gamma_s}) + \nabla\mathbb{A}_\lambda^{-1}[v_2^*+\lambda v_1^*], \nabla D_\lambda(\phi|_{\Gamma_s})\right)_{\Omega_s}, \\
\qquad \text{for all } \phi \in \mathbf{H}^1_{\Gamma_f}(\Omega_f).
\end{array}
$$

This solution of this abstract system follows from the Babuška-Brezzi Theorem, inasmuch as the *inf-sup* condition is satisfied: That is, we show there exists $\beta > 0$ such that $\forall \rho \in L^2(\Omega_f)$,

$$\sup_{\phi \in \mathbf{H}^1_{\Gamma_f}(\Omega_f)} \frac{-(\rho, div(\phi))_{\Omega_f}}{\|\phi\|_{1,\Omega_f}} \geq \beta \|\rho\|_{\Omega_f}.$$

Note that coupled variational relation lends immediately itself to a FEM approximation in which the finite dimensional "stiffness" matrix equation takes the form

$$\begin{bmatrix} A_\lambda & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u_h \\ p_h \end{bmatrix} = F_h,$$

where

$$u_h \in V_h \subset \mathbf{H}^1_{\Gamma_f}(\Omega_f), \quad p_h \in \Pi_h \subset L^2(\Omega_f),$$

where $\{V_h, \Pi_h\}$ are finite dimensional (FEM) subspaces (and h is a parameter of discretization, or in the mesh below, for the fluid geometry, the length of each element hypotenuse).

The nonsingularity of this matrix is equivalent to establishing the *discrete inf-sup condition*. That is, there exists a $\beta > 0$ such that, *uniformly in h*,

$$\sup_{\phi_h \in V_h} \frac{-\int \rho_h div(\phi_h)}{\|\phi_h\|_{1,\Omega_f}} \geq \beta \|\rho_h\|_{\Omega_f} \quad \text{for all } \rho_h \in \Pi_h.$$

Theorem

([G. A. and M. Dvorak, 2009]). If $V_h \subset \mathbf{H}^1_{\Gamma_f}(\Omega_f)$ are piecewise quadratic functions, and $\Pi_h \subset L^2(\Omega_f)$ are piecewise linears, than the discrete inf-sup condition is satisfied.

Remarks.

- This validity of the discrete inf-sup condition allows one to eventually obtain convergence estimates for the fluid *and* structural components.
- Ingredients from the work done for P_2/P_1 mixed finite elements for *uncoupled* Stokes flow -under the *no-slip* boundary condition - are certainly used here. ({Babuška, Brezzi, Brezzi-Fortin, 1973-}, [Glowinski-Pironneau, 1979], [Bercovier-Pironneau, 1979], [Ern-Guermond, 2004], [Verfürth, 1984], and many others).

However, the fact the fluid does not vanish on the boundary interface presents novelties not seen in uncoupled fluid problems.

For example: In the course of establishing the *inf-sup condition* which allows for an invocation of Babuška Brezzi, one must justify that given any $\rho \in L^2(\Omega_f)$, there exists an $\eta \in \mathbf{H}^1_{\Gamma_f}(\Omega_f)$ to the following BVP (see [G.A. and R. Triggiani, 2009] :

$$\begin{aligned} div(\eta) &= -\rho \text{ in } \Omega_f \\ \eta|_{\Gamma_f} &= 0 \text{ on } \Gamma_f \\ \eta|_{\Gamma_s} &= -\frac{\left(\int \rho d\Omega_s\right)}{meas(\Gamma_s)}\nu \text{ on } \Gamma_s. \end{aligned}$$

(In particular, the data of the problem satisfies a compatibility condition necessary for a solution).

Implicit in this work however is an underlying assumption that Γ_s is smooth enough so that the normal vector has enough Sobolev regularity. But what if the geometry is a polygonal, the usual setting for the standard FEM?
Then to establish the inf-sup condition on such nonsmooth domains, we use a particular function found in Grisvard's book of 1986: I.e.: If the geometry Ω_f is Lipschitz there exists a $\mu \in [C^\infty(\bar{\Omega}_f)]^n$ and $\delta > 0$ such that $\mu \cdot \nu > 0$ on $\partial\Omega_f$.

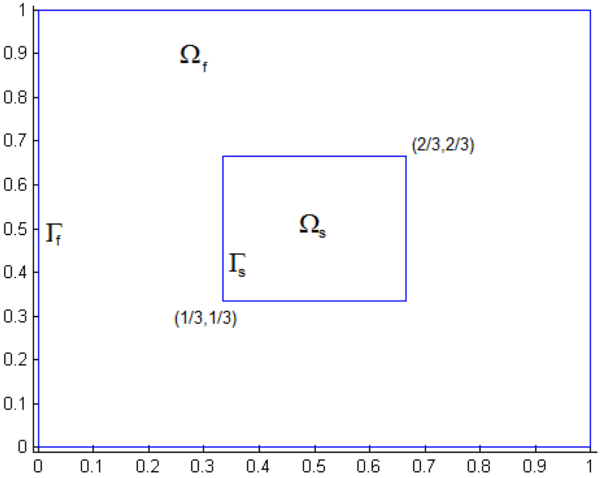
Therewith, we can establish the inf-sup condition by having, for given $\rho \in L^2(\Omega_f)$, $\omega \in \mathbf{H}^1_{\Gamma_f}(\Omega_f)$ to be the solution of

$$\begin{aligned} div(\omega) &= -\rho \langle \mu, \nu \rangle_{\Gamma_s} \text{ in } \Omega_f \\ \omega|_{\Gamma_f} &= 0 \text{ on } \Gamma_f \\ \omega|_{\Gamma_s} &= \left(\int \rho d\Omega_s \right) \mu \text{ on } \Gamma_s. \end{aligned}$$

Another complication in establishing the discrete inf-sup condition, coming from the fluid-structure model, is the lack of (static) H^2-regularity for the fluid component of the solution. In consequence, the classic (nodal) interpolant is not valid, and polynomial interpolants developed for non-smooth functions must be invoked (e.g., [Clément, 1975], [Scott-Zhang, 1990], [Bernardi-Girault, 1998]).

A Numerical Example

Let solid domain $\Omega_s = (1/3, 2/3)^2$, and fluid domain $\Omega_f = (0,1)^2 \setminus [1/3, 2/3]^2$.



On this geometry, we consider the problem of finding $[f, v_1, v_2] \in D(\mathbf{A})$ which solves

$$(\lambda I - \mathbf{A}) \begin{bmatrix} f \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} f^* \\ v_1^* \\ v_2^* \end{bmatrix},$$

for given data $[f^*, v_1^*, v_2^*] \in \mathbf{H}$, and fixed $\lambda > 0$.
For the sake of having an explicitly computable problem, set

$$\begin{bmatrix} f^* \\ v_1^* \\ v_2^* \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ \eta \\ 0 \end{bmatrix};$$

here, η is the unique solution of the following elliptic problem:

$$\Delta\eta - \eta = 0 \ \text{ in } \Omega_s; \ \ \frac{\partial\eta}{\partial\nu} = \nu \ \text{ on } \Gamma_s.$$

It is known that $[0, \eta, 0]$ is an eigenfunction corresponding to the eigenvalue *zero* of the fluid-structure generator $\mathbf{A} : D(\mathbf{A}) \subset \mathbf{H} \rightarrow \mathbf{H}$ (G.A., R. Triggiani, 2006]). So the actual solution can be explicitly computed for the given canonical geometry Ω_s. Moreover, from the boundary conditions, we see immediately that the pressure function corresponding to data $[0, \eta, 0]$ is $\pi = -1$.

Applying the said FEM mixed method with P_2/P_1 elements, we obtain the following error:

No. of Elements	$\|f_h - f\|_{\Omega_f}$	$\|\pi_h - \pi\|_{\Omega_f}$
72	$8.86 \cdot 10^{-5}$	$2.41 \cdot 10^{-4}$
288	$2.39 \cdot 10^{-5}$	$6.28 \cdot 10^{-5}$

No. of Elements	$\|(v_1)_h - v_1\|_{1,\Omega_s}$	$\|(v_2)_h - v_2\|_{\Omega_s}$
72	$1.60 \cdot 10^{-5}$	$3.61 \cdot 10^{-5}$
288	$4.86 \cdot 10^{-6}$	$7.86 \cdot 10^{-6}$

Note that the given error is *exactly* of the order expected from the FEM analysis, no more and no less.