Semigroup Growth Bounds

First Meeting on Asymptotics of Operator Semigroups

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Norms of Semigroup Operators

There are several distinct issues for a one-parameter semigroup $T_t = \mathrm{e}^{At}$ acting in a Banach space $\mathcal{B}$.

- The long time asymptotics of $\| T_t \|$;
- The short time asymptotics of $\| T_t \|$;
- The intermediate time behaviour of $\| T_t \|$;
- The spectrum of A;
- The behaviour of the norms of the resolvent operators
  $R_z = (zI - A)^{-1}$.

The Classical Bounds

Every semigroup has a bound of the form

$$\|T_t\| \le M e^{at} \qquad \text{for all } t \ge 0.$$

This implies that

$$\|R_z\| \le M(\mathrm{Re}\,(z) - a)^{-1}$$

for all z satisfying $\mathrm{Re}\,(z) > a$.

The precise form of the converse was proved by Feller, Miyadera and Phillips.

Dissipativity

If $\mathcal{B}$ is a Hilbert space and

$$\mathrm{Re}\,\langle Af, f\rangle \leq a$$

for all $f \in \mathrm{Dom}(A)$ then

$$\|T_t\| \leq \mathrm{e}^{at} \qquad \text{for all } t \geq 0$$

and conversely.

The Asymptotic Growth Rate

The infimum of all possible a is given by

$$\omega_0 = \lim_{t \to +\infty} t^{-1} \log(\| T_t \|).$$

This implies that

$$\mathrm{Spec}(A) \subseteq \{z : \mathrm{Re}\,(z) \le \omega_0\}$$

and

$$\| T_t \| \ge \mathrm{e}^{\omega_0 t} \text{ for all } t > 0.$$

Zabczyk's Example[1]

$$\mathrm{Spec}(T_t) \supseteq \left\{ \mathrm{e}^{zt} : z \in \mathrm{Spec}(A) \right\}.$$

but the two sides need not be equal.

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There exists a one-parameter group T_t acting in a Hilbert space $\mathcal{H}$ such that

$$\mathrm{Spec}(A) \subseteq i\mathbf{R}$$

but

$$\| T_t \| = \mathrm{e}^{|t|} \qquad \text{for all } t \in \mathbf{R}.$$

[1] Zabczyk 1975

The Schrödinger Group[2]

The operators $T_t = \mathrm{e}^{i\Delta t}$ are unbounded on $L^p(\mathbf{R}^n)$ for all $p \neq 2$ and $0 \neq t \in \mathbf{R}$ in spite of the fact that

$$\mathrm{Spec}(\Delta) \subseteq \mathbf{R}.$$

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The resolvents of Δ satisfy

$$\|R_z\| \leq c_p |\mathrm{Im}\,(z)|^{-1}$$

for all $z \notin \mathbf{R}$, where $c_p \to 1$ as $p \to 2$.

[2] Hormander 1960

An Indefinite ODE[3]

If $\mathcal{H} = L^2(-\pi, \pi)$ and $0 < \varepsilon < 2$ and

$$(Lf)(\theta) = \varepsilon \frac{\mathrm{d}}{\mathrm{d}\theta} \left\{ \sin(\theta) \frac{\mathrm{d}f}{\mathrm{d}\theta} \right\} + \frac{\mathrm{d}f}{\mathrm{d}\theta}$$

then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = Lf(t)$$

describes the evolution of a thin fluid layer inside a rotating cylinder.

[3] Benilov, O'Brien, Sazonov, Weir et al. 2000-2008

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If $0 < \varepsilon < 2$ then L has purely imaginary spectrum consisting of a discrete sequence of eigenvalues.

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If $0 < \varepsilon < 2$ then L has purely imaginary spectrum consisting of a discrete sequence of eigenvalues.

If $\varepsilon > 2$ then the spectrum of L includes the entire imaginary axis and probably the entire complex plane.

If $0 < \varepsilon < 2$ the resolvent operators are all compact but e^{Lt} is unbounded for all $t \neq 0$.

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An Example with an Oscillating Norm

Let

$$(T_t f)(x) = \frac{a(x+t)}{a(x)} f(x+t)$$

for all $f \in L^2(0,\infty)$ and all $t \geq 0$.

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If $c > 1$ then the choice

$$a(x) = 1 + (c-1)\sin^2(\pi x/2)$$

leads to $\|T_{2n}\| = 1$ and $\|T_{(2n+1)}\| = c$ for all positive integers n.

Study of N(t)

Definition of $N(t)$[5]

We define $N(t)$ to be the upper log-concave envelope of $\|T_t\|$.

In other words $\nu(t) = \log(N(t))$ is defined to be the smallest concave function satisfying $\nu(t) \geq \log(\|T_t\|)$ for all $t \geq 0$.

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It is immediate that $N(t)$ is continuous for $t > 0$, and that

$$1 = N(0) \leq \lim_{t \to 0+} N(t).$$

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Normalization

We replace T_t by $T_t \mathrm{e}^{-\omega_0 t}$ or, equivalently, normalize our problem by assuming that $\omega_0 = 0$.

This implies that $\mathrm{Spec}(A) \subseteq \{z : \mathrm{Re}\,(z) \leq 0\}$.

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$N(t)$ is increasing function of t but it increases sub-exponentially as $t \to +\infty$.

The Legendre Transform

We study the function $N(t)$ via a transform, defined for all $\omega > 0$ by

$$M(\omega) = \sup\{\| T_t \| \mathrm{e}^{-\omega t} : t \geq 0\}.$$

$M(\omega)$ is a monotonic decreasing function of ω which satisfies

$$\lim_{\omega \to +\infty} M(\omega) = \limsup_{t \to 0} \| T_t \|.$$

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$$N(t) = \inf\{M(\omega) e^{\omega t} : 0 < \omega < \infty\}$$

for all $t > 0$ by the theory of the Legendre transform.

Theorem

If $a > 0$, $b \in \mathbf{R}$ *and* $a\|R_{a+ib}\| = c \geq 1$ *then*

$$M(\omega) \geq \tilde{M}(\omega) := \begin{cases} (a-\omega)c/a & \text{if } 0 < \omega \leq r = a(1-1/c) \\ 1 & \text{otherwise.} \end{cases}$$

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Proof.

The formula

$$R_{a+ib} = \int_0^\infty T_t \mathrm{e}^{-(a+ib)t} \, \mathrm{d}t$$

implies that

$$c/a \leq \int_0^\infty N(t)\mathrm{e}^{-at} \, \mathrm{d}t \leq \int_0^\infty M(\omega)\mathrm{e}^{\omega t - at} \, \mathrm{d}t = M(\omega)(a-\omega)^{-1}$$

for all ω such that $0 < \omega < a$. □

The following theorem implies that if the resolvent norm is significantly larger than $1/a$ for some large a then $N(t)$ must grow rapidly for small $t > 0$.

Theorem

If $a\|R_{a+ib}\| = c \geq 1$ *and* $r = a(1 - 1/c)$ *then*

$$N(t) \geq \min\{\mathrm{e}^{rt}, c\}$$

for all $t \geq 0$.

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Proof.

This uses

$$N(t) = \inf\{M(\omega)\mathrm{e}^{\omega t} : \omega > 0\} \geq \inf\{\tilde{M}(\omega)\mathrm{e}^{\omega t} : \omega > 0\}.$$

□

Schrodinger operators

Schrödinger Operators with Non-Negative Potentials

Theorem

Let $H = -\Delta + V$, *acting in* $L^1(\mathbf{R}^n)$, *where* $V \geq 0$. *Then*

$$(\mathrm{e}^{-Ht}f)(x) = \int_{\mathbf{R}^n} K(t,x,y)f(y)\,\mathrm{d}y$$

where

$$0 \leq K(t,x,y) \leq (4\pi t)^{-n/2}\mathrm{e}^{-|x-y|^2/4t}.$$

This can be proved by functional integration or the Trotter product formula.

Theorem

Let $H = -\Delta + V$*, acting in* $L^1(\mathbf{R}^n)$*, where* V *is continuous and bounded below, with*

$$c = -\inf\{V(x) : x \in \mathbf{R}^N\}.$$

Then

$$c = \min\{\omega : \|\mathrm{e}^{-Ht}\| \leq \mathrm{e}^{\omega t} \text{ for all } t \geq 0\}.$$

Note that the situation is quite different in $L^2(\mathbf{R}^n)$.

Polynomial Growth of L^1 Norms

Theorem (Murata 1984, 1985 and Davies-Simon 1991.)

Let $N \geq 3$. There exists a Schrödinger semigroup e^{-Kt} acting in $L^1(\mathbf{R}^N)$ and positive constants c_1, c_2, σ_1 and σ_2 such that

$$c_1(1+t)^{\sigma_1} \leq \|\mathrm{e}^{-Kt}\| \leq c_2(1+t)^{\sigma_2}$$

for all $t \geq 0$, even though K is non-negative considered as an operator acting in $L^2(\mathbf{R}^N)$.

The constants σ_1 and σ_2 are more or less equal.

The proof involves zero energy resonances.

The Explicit Example

The potential is given by

$$V(x) = \begin{cases} -c|x|^{-2} & \text{if } |x| \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

where

$$0 < c < \frac{(n-2)^2}{4}.$$

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The zero energy resonance is of the form

$$0 < \eta(x) = \begin{cases} |x|^{-\alpha_1} - \beta\,|x|^{-\alpha_2} & \text{if } |x| \geq 1 \\ 1 - \beta & \text{otherwise} \end{cases}$$

for certain positive constants α_1, α_2 and β.