Random Matrices from the Classical Compact Groups: a Panorama

Part V: Exact Formulas for Eigenvalue Distributions

Mark Meckes

Case Western Reserve University

Oxford, 22 February 2021

Eigenvalues of unitary matrices

Basic observations:

- If λ is an eigenvalue of $U \in \mathbb{U}(n)$ then $|\lambda| = 1$.
- If λ is an eigenvalue of $U \in \mathbb{O}(n)$ or $\mathbb{Sp}(n)$ then so is $\overline{\lambda}$.
- $\lambda = 1$ is an eigenvalue of every $U \in \mathbb{SO}(2n+1)$,
  $\lambda = -1$ is an eigenvalue of every $U \in \mathbb{SO}^{-}(2n+1)$, and
  $\lambda = \pm 1$ are both eigenvalues of every $U \in \mathbb{SO}^{-}(2n)$.

We call ± 1 the trivial eigenvalues and often focus on the nontrivial ones.

Eigenvalues of unitary matrices

Basic notations:

- Eigenangles:
  - For $U \in \mathbb{U}(n)$, we denote the eigenvalues as $e^{i\theta_j}$ for $-\pi < \theta_j < \pi$, $1 \le j \le n$.
  - For $U \in \mathbb{SO}(2n)$, $\mathbb{SO}(2n+1)$, $\mathbb{SO}^-(2n+1)$, $\mathbb{SO}^-(2n+2)$ or $\mathbb{Sp}(n)$, we denote the eigenvalues with $\operatorname{Im} \lambda > 0$ as $e^{i\theta_j}$ for $0 < \theta_j < \pi$, $1 \le j \le n$.

- Counting function: For $A \subseteq \mathbb{R}$, $\mathcal{N}_A = \# \left\{ j \middle| \theta_j \in A \right\}$.

- Spectral measure: For $U \in \mathbb{U}(n)$, $\mu_U = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j}$.

The Weyl integration formula for $\mathbb{U}(n)$

Theorem (Weyl)

Let $U \in \mathbb{U}(n)$ be random. The joint density of the unordered eigenangles of U is

$$\frac{1}{n!(2\pi)^n} \prod_{1 \le j < k \le n} \left| e^{i\theta_j} - e^{i\theta_k} \right|^2 .$$

That is, if $f : \mathbb{U}(n) \to \mathbb{C}$ depends only on the unordered eigenvalues of U, then

$$\mathbb{E} f(U) = \frac{1}{n!(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{1 \le j < k \le n} \left| e^{i\theta_j} - e^{i\theta_k} \right|^2 \times f\big(\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})\big)\, d\theta_1 \cdots d\theta_n.$$

The Weyl integration formula for $\mathbb{U}(n)$

Remarks on Weyl's formula:

- Weyl's formula expresses an integral of a simple function on a complicated space as an integral of a more complicated function on a simple space.
- It's a beginning and not an end!
- The proof is Lie-theoretic, based on the invariance property

$$f(D) = f(UDU^*)$$

for fixed $D, U \in \mathbb{U}(n)$.
- The density is essentially the Jacobian determinant for

$$(\mathbb{U}(n)/\mathbb{T}) \times \mathbb{T} \to \mathbb{U}(n), \qquad (U\mathbb{T}, D) \mapsto UDU^*,$$

where $\mathbb{T} = \{D \in \mathbb{U}(n) | D \text{ is diagonal}\}$ is the maximal torus in $\mathbb{U}(n)$.

Circular ensembles

A collection of random points $\{e^{i\theta_j}\}$ with density $\propto \prod_{1 \leq j < k \leq n} \left| e^{i\theta_j} - e^{i\theta_k} \right|^\beta$ is called the circular β-ensemble, and models a 1-dimensional gas with logarithmic potential at inverse temperature β.

Special values of β are related to symmetries considered in quantum mechanics:

- $\beta = 2$: Circular Unitary Ensemble — eigenvalues of random $U \in \mathbb{U}(n)$.
- $\beta = 1$: Circular Orthogonal Ensemble — eigenvalues of $U^T U$ for random $U \in \mathbb{U}(n)$.
- $\beta = 4$: Circular Symplectic Ensemble.

Other Weyl integration formulas

Similar formulas exist for the other cases, e.g.:

Theorem (Weyl)

Let $U \in \mathbb{SO}(2n+1)$ be random. The joint density of the unordered nontrivial eigenangles of U is

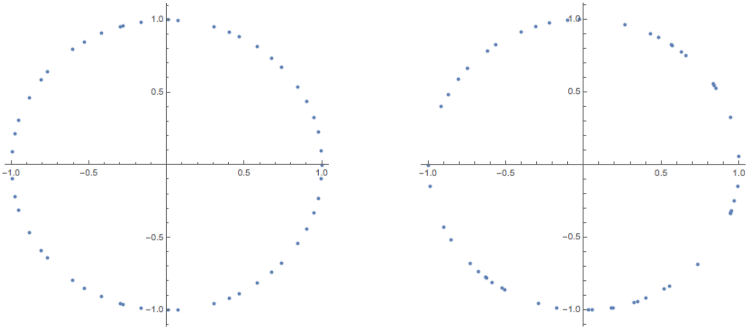
$$\frac{2^n}{n!\pi^n} \prod_{j=1}^{n} \sin^2\left(\frac{\theta_j}{2}\right) \prod_{1 \le j < k \le n} (2\cos\theta_j - 2\cos\theta_k)^2.$$

Fun fact: The eigenangles for $\mathbb{Sp}(n)$ have the same distribution as the nontrivial eigenangles for $\mathbb{SO}^-(2n+2)$.

Eigenvalue repulsion

The density is small whenever two eigenvalues (trivial or not) are close to each other.

The eigenvalues repel each other!



Eigenvalues of $U \in \mathbb{SO}(51)$ versus 51 independent uniform random points.

Joint intensities / correlation functions

The joint intensities (or correlation functions) of $\{\theta_j\}$ are defined by

$$\mathbb{E} \prod_{j=1}^{k} \mathcal{N}_{A_j} = \int_{\prod_j A_j} \rho_k(x_1, \dots, x_k)\, dx_1 \dots dx_k$$

whenever $A_1, \dots, A_k \subseteq \mathbb{R}$ are disjoint.

$\rho_k(x_1, \dots, x_k) \approx$ the likelihood of finding one eigenangle near each of $x_1, \dots, x_k$.

Determinantal point processes

Theorem (Dyson)

If $U \in \mathbb{U}(n)$ is random then the joint intensities of the eigenangles are

$$\rho_k(x_1, \ldots, x_k) = \det\left[K_n(x_i, x_j)\right]_{i,j=1}^{k},$$

where

$$K_n(x, y) = \frac{1}{2\pi} \sum_{j=0}^{n-1} e^{ij(x-y)} = \frac{\sin\left(\frac{n}{2}(x-y)\right)}{2\pi \sin\left(\frac{1}{2}(x-y)\right)}.$$

We say that $\{\theta_j\}$ is a determinantal point process on $[-\pi, \pi]$ with kernel K_n.

Similar versions hold for nontrivial eigenangles (with different kernels K_n) for all the other cases discussed today (Katz–Sarnak).

Determinantal point processes

One of the wonderful things about determinantal processes:

Theorem (Hough–Krishnapur–Peres–Virág)

Suppose $\{\theta_j\}$ *is a DPP on* $\mathbb{R}$ *with Hermitian kernel* K*.*
Given $A \subseteq \mathbb{R}$*, let* $\{\alpha_j\} \subseteq [0,1]$ *be the eigenvalues of the integral operator* T_A *on* $L^2(A)$ *with kernel* K*:*

$$Tf(x) = \int_A K(x,y) f(y)\, dy.$$

Then

$$\mathcal{N}_A \stackrel{D}{=} \sum_j \varepsilon_j,$$

where $\{\varepsilon_j\}$ *are independent Bernoulli random variables with* $\mathbb{P}[\varepsilon_j = 1] = \alpha_j$*.*

Toeplitz determinants

Theorem (Heine–Szegő formula)

If $f(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ *and* $U \in \mathbb{U}(n)$ *is random with eigenvalues* $\{\lambda_j\}$, *then*

$$\mathbb{E}\prod_{j=1}^{n} f(\lambda_j) = \det \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_{-1} & a_0 & \ddots & & a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{-(n-2)} & & \ddots & a_0 & a_1 \\ a_{-(n-1)} & a_{-(n-2)} & \cdots & a_{-1} & a_0 \end{bmatrix}.$$

The latter is a Toeplitz determinant and asymptotics as $n \to \infty$ are well understood (Szegő limit theorem).

Traces of powers

Theorem (Diaconis–Shahshahani)

Suppose that $U \in \mathbb{U}(n)$ is random and Z is a standard complex random variable. Let $k \in \mathbb{N}$ and $p, q \in \mathbb{N} \cup \{0\}$. Then

$$\mathbb{E}(\operatorname{Tr} U^k)^p (\overline{\operatorname{Tr} U^k})^q = \delta_{pq} k^p p! = \mathbb{E}(\sqrt{k}Z)^p (\overline{\sqrt{k}Z})^q$$

whenever $n \geq \max\{kp, kq\}$.

Note $\operatorname{Tr} U^k = \sum_{j=1}^n \lambda_j^k =: p_k(\lambda_1, \ldots, \lambda_n)$ (power sum).

So $\operatorname{Tr} U^k$ is distributed remarkably similarly to $\sqrt{k}Z$.

Moreover: The joint moments of $\operatorname{Tr} U, \operatorname{Tr} U^2, \ldots, \operatorname{Tr} U^k$ match those of independent complex normals for sufficiently large n.

Traces of powers

Ingredients in the proof:

- Products of power sums are symmetric functions.
- The space of symmetric functions is spanned by Schur polynomials, and the change of basis can be computed exactly using representation theory of symmetric groups.
- Traces of Schur polynomials are irreducible characters of $\mathbb{U}(n)$.

Will all this, the proof reduces to some fairly straightforward combinatorics.

Versions for $\mathbb{O}(n)$ and $\mathbb{Sp}(n)$ are also known, but the representation theory becomes substantially more difficult.

Traces and increasing subsequences

For a permutation $\pi \in S_k$, let $\ell(\pi)$ be the length of the longest increasing subsequence of π.

Theorem (Rains)

1. *If $U \in \mathbb{U}(n)$ is random, then $\mathbb{E}\left|\operatorname{Tr} U\right|^{2k}$ is the number of $\pi \in S_k$ with $\ell(\pi) \le n$.*
2. *If $U \in \mathbb{O}(n)$ is random, then $\mathbb{E}(\operatorname{Tr} U)^k$ is the number of $\pi \in S_k$ such that $\pi^{-1} = \pi$, π has no fixed points, and $\ell(\pi) \le n$.*
3. *If $U \in \mathbb{S}\mathrm{p}(n)$ is random, then $\mathbb{E}(\operatorname{Tr} U)^k$ is the number of $\pi \in S_k$ such that $\pi^{-1} = \pi$, π has no fixed points, and $\ell(\pi) \le 2n$.*

More complicated similar results hold for U^m.

Spectra of powers of random matrices

Theorem (Rains)

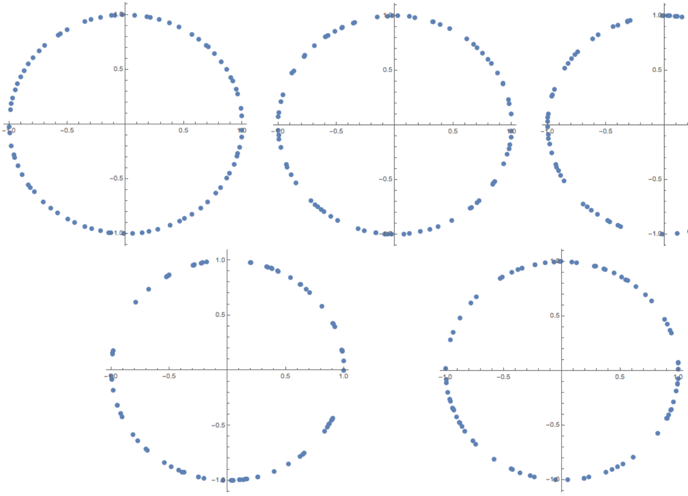
If $U \in \mathbb{U}(n)$ is random and $1 \leq m \leq n$, then the collection of eigenvalues of U^m has the same distribution as m independent copies of the eigenvalues of random matrices in $\mathbb{U}(n/m)$.

(This should be modified in an almost-obvious way if n is not a multiple of m.)

If $m \geq n$, the collection of eigenvalues of U^m is distributed like n independent uniform random points in the unit circle.

It can be modified in a less obvious way for other groups.

Spectra of powers of random matrices



Eigenvalues of U^m for $U \in \mathbb{U}(80)$ and $m = 1, 5, 20, 45, 80$.

Additional references

- Daniel Bump, *Lie Groups*, 2nd edition, Springer, 2013.
- Daniel Bump and Persi Diaconis, "Toeplitz minors", *Journal of Combinatorial Theory Series A* 97, pp. 252–271, 2002.